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Extremal Black Hole and Flux Vacua Attractors

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Abstract

These lectures provide a pedagogical, introductory review of the so-called Attractor Mechanism (AM) at work in two different 4-dimensional frameworks: extremal black holes in $\mathcal{N}=2$ supergravity and $\mathcal{N}=1$ flux compactifications. In the first case, AM determines the stabilization of scalars at the black hole event horizon purely in terms of the electric and magnetic charges, whereas in the second context the AM is responsible for the stabilization of the universal axion-dilaton and of the (complex structure) moduli purely in terms of the RR and NSNS fluxes. Two equivalent approaches to AM, namely the so-called "criticality conditions" and "New Attractor" ones, are analyzed in detail in both frameworks, whose analogies and differences are discussed. Also a stringy analysis of both frameworks (relying on Hodge-decomposition techniques) is performed, respectively considering Type IIB compactified on CY_3 and its orientifolded version, associated with $\frac{CY_3\times T^2}{\mathbb{Z}_2}$. Finally, recent results on the U-duality orbits and moduli spaces of non-BPS extremal black hole attractors in $3\leqslant\mathcal{N}\leqslant 8$, $d=4$ supergravities are reported.

Contents

1 Introduction

After the original papers [1]-[5] from mid 90's dealing mostly with the Bogomol'ny-Prasad-Sommerfeld (BPS) black holes (BHs), *extremal BH attractors* have been recently widely investigated [6]–[53] (see also [54]–[62]). Such a *renaissance* is mainly due to the (re)discovery of new classes of solutions to the attractor equations corresponding to non-BPS horizon geometries: certain configurations of moduli stabilized near the horizon of extremal BHs exist which break supersymmetry. In addition, the stabilization of moduli in the context of string theory has become a central issue of string cosmology. The attractor equations used in the past for stabilizing moduli near the horizon of an extremal BH have turned out to be useful in the context of flux vacua.

In this introduction we will first briefly remind the basic structure of the BPS BH attractors in $\mathcal{N}=2$, $d=4$ supergravity. After that, we will outline the main features of the recent developments in non-BPS extremal BH attractors and flux vacua, a detailed description of which will be given in the subsequent sections.

An horizon extremal BH attractor geometry is in general supported by particular configurations of the $1\times(2n_V+2)$ symplectic vector of the BH field-strength fluxes, *i.e.* of the BH magnetic and electric charges:

$$Q\equiv\left(p^\Lambda,q_\Lambda\right),\qquad p^\Lambda\equiv\frac{1}{4\pi}\int_{S^2_\infty}\mathcal{F}^\Lambda,\quad q_\Lambda\equiv\frac{1}{4\pi}\int_{S^2_\infty}\mathcal{G}_\Lambda,\quad \Lambda=0,1,...,n_V,\tag{1.1}$$

where, in the case of $\mathcal{N}=2$, $d=4$ supergravity, n_V denotes the number of Abelian vector supermultiplets coupled to the supergravity one (containing the Maxwell vector A^0, usually named *graviphoton*). Here $\mathcal{F}^\Lambda=dA^\Lambda$ and $\mathcal{G}_\Lambda$ is the "dual" field-strength two-form [63, 64].

BPS BH attractor equations fix the values of all moduli near BH horizon in terms of the electric and magnetic charges. The most compact form of these equations was given in [65], where the Kähler invariant period $\left(Y^\Lambda,F_\Lambda\left(Y\right)\right)$ was introduced by multiplying the covariantly holomorphic period $V(z,\bar{z})$ (see Eq. (2.4) below) on the (complex conjugate of the) $\mathcal{N}=2$, $d=4$ *central charge function* $\bar{Z}$, so that

$$\bar{Z}V\equiv\left(Y^\Lambda,F_\Lambda\left(Y\right)\right)\tag{1.2}$$

where $Y^\Lambda=Y^\Lambda\left(z,\bar{z}\right)$ and $V=\left(L^\Lambda,M_\Lambda\right)$. In terms of such variables, the BPS attractor equations are very simple and state that at the BH horizon the moduli $(z,\bar{z})$ depend on electric and magnetic charges so that equations

$$Y^\Lambda-\bar{Y}^\Lambda=ip^\Lambda\ ,\qquad F_\Lambda\left(Y\right)-\bar{F}_\Lambda\left(\bar{Y}\right)=iq_\Lambda\tag{1.3}$$

are satisfied, and their solution defines moduli as functions of charges

$$z_{cr}=z_{cr}(p,q)\,,\qquad \bar{z}_{cr}=\bar{z}_{cr}(p,q).\tag{1.4}$$

BPS attractors equations (1.3) are equivalent to the condition of unbroken supersymmetry: $DZ=0$.

A simple way to derive the BH attractor equations, which also gives a clear link to their use in the context of flux vacua, is by using the language of string theory compactified on a Calabi-Yau threefold (CY_3) [66]. One starts with the Hodge decomposition of the 3-form flux (see Eq. (3.4.1.30) below)

$$\mathcal{H}_3=-2Im[\bar{Z}\hat{\Omega}_3-\bar{D}^i\bar{Z}D_i\hat{\Omega}_3]=\int_{S^2_\infty}\widehat{\mathcal{F}}^+,\tag{1.5}$$

where $\hat{\Omega}_3$ is the covariantly holomorphic 3-form of the CY_3, $\widehat{\mathcal{F}}^+$ is the self-dual 5-form of type IIB string theory and S^2_∞ is the 2-sphere at infinity, as in the definition (1.1) (see *e.g.* [64]). By integration over a symplectic basis of 3-cycles of CY_3 the decomposition (1.5) can be brought to the form (see Eq. (3.3.1.10) below)

$$Q^T = -2Im[\bar{Z}V - \bar{D}^i\bar{Z}D_iV]. \tag{1.6}$$

By inserting the condition of unbroken supersymmetry $D_iZ = 0$ into the identities (1.5) and (1.6), one obtains the BPS extremal BH attractor Eqs. (1.3) in a stringy framework:

$$\mathcal{H}_3 = -2Im[\bar{Z}\hat{\Omega}_3]_{DZ=0}, \tag{1.7}$$

or equivalently:

$$Q^T = -2Im\left[\overline{Z}V\right]_{DZ=0}. \tag{1.8}$$

This attractor Eq. presents a particular case of the criticality condition for the so-called effective BH potential, $\partial_iV_{BH} = 0$, where (see definition (3.1.1) below)

$$V_{BH}(z,\bar{z}) \equiv |Z|^2 + g^{i\bar{j}}\left(D_iZ\right)\overline{D}_{\bar{j}}\overline{Z}. \tag{1.9}$$

Another important feature of the BPS attractors is the relation between the second derivative of V_{BH} at the critical points $\partial V_{BH} = 0$ and the metric $g_{i\bar{j}}$ of the scalar manifold (usually called *moduli space* in string theory), namely

$$\left(\partial_i\overline{\partial}_{\bar{j}}V_{BH}\right)_{\partial V_{BH}=0} = 2\left(g_{i\bar{j}}V_{BH}\right)_{\partial V_{BH}=0}. \tag{1.10}$$

Since V_{BH} at the supersymmetric critical point $DZ = 0$ (with non-vanishing entropy) is strictly positive ($V_{BH}|_{DZ=0} = |Z|^2_{DZ=0} > 0$), Eq. (1.10) implies that all BPS attractors are stable, at least as long as the metric of the moduli space is strictly positive definite. Note that in the BPS case the condition of non-vanishing entropy requires $Z|_{DZ=0} \neq 0$.

The recent developments with *non-BPS BH attractors* can be described shortly as follows. For extremal non-BPS BH solutions of $\mathcal{N} = 2$, $d = 4$ supergravity one finds the mechanism of stabilization of moduli near BH horizon with some properties of the same nature as in BPS case, and some properties somewhat different.

Many nice features of the BH attractors in the past were associated with the unbroken supersymmetry of BPS BHs. During the last few years the basic reason for the attractor behaviour of extremal BHs has been discovered to be geometrical[1]: extremal BHs (regardless their supersymmetry-preserving features) all have moduli which acquire fixed values at the BH horizon independent of their values at infinity! Their values at the horizon depend only on the electric and magnetic BH charges. The existence of an infinite throat in the space-time geometry of extremal BHs leads to an evolution towards the horizon such that the moduli forget their initial conditions at (spatial) infinity [14]. Since a Schwarzschild-type BH geometry with non-vanishing horizon area is never extremal, this phenomenum never takes place when solving the equations of motion for scalar fields in such a background: their values at the horizon depend on the initial conditions of the radial dynamical evolution, because there are no coordinate systems with infinite distance from the event horizon.

[1]We are grateful to A. Linde for this insight, see also [14].

A simple qualifier of both BPS and non-BPS attractors remains valid in the form of a critical point of the BH potential:

$$\partial V_{BH} = 0 \ : \qquad for\ BPS: \quad DZ = 0, \qquad for\ non-BPS: \quad DZ \neq 0. \tag{1.11}$$

The non-BPS attractor Eqs. in the form generalizing Eqs. (1.3) can be given separately for the cases $Z \neq 0$ and $Z = 0$ [2].

In the case $Z \neq 0$ one finds (see Eq. (3.3.2.3) below)

$$Q^T = -2Im\left\{\left[\overline{Z}V - \frac{i}{2}\frac{Z}{|Z|^2}C^{\overline{i}\overline{j}\overline{k}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)\left(\overline{D}_{\overline{k}}\overline{Z}\right)\overline{D}_{\overline{i}}\overline{V}\right]\right\}_{non-BPS, Z\neq 0}. \tag{1.12}$$

Here one starts with the identity (1.6) and replaces the second term using the expression for it derived from the non-BPS $Z \neq 0$ criticality condition $\partial V_{BH} = 0$. The attractor Eq. (1.12) is a clear generalization of the BPS attractor Eq. (1.3)-(1.8) with $Z \neq 0$: at $DZ = 0$ the second term in the right-hand side (r.h.s.) of Eq. (1.12) vanishes and it reduces exactly to Eq. (1.8) or its detailed form given by Eq. (1.3).

For both classes ($Z \neq 0$ and $Z = 0$) of non-BPS attractors the critical value of V_{BH} remains positive, since by definition V_{BH} is a real, positive function in the scalar manifold. However, the universal BPS stability condition (1.10) is not valid anymore and one has to study this issue separately[3].

In the present review we will consider only critical points of V_{BH} ($\frac{1}{2}$-BPS as well as non-BPS) which are *non-degenerate*, *i.e.* with a finite, non-vanishing horizon area, corresponding to the so-called *"large"* BHs[4].

Due to the so-called Attractor Mechanism (AM) [1]-[5], the Bekenstein-Hawking entropy [67] of *"large"* extremal BHs can be obtained by extremizing $V_{BH}(\phi, Q)$, where "ϕ" now denotes the set of real scalars relevant for the AM, and Q is defined by Eq. (1.1). In $\mathcal{N} = 2$, $d = 4$ supergravity, non-degenerate attractor horizon geometries correspond to BH solitonic states belonging to $\frac{1}{2}$-BPS "short massive multiplets" or to non-BPS "long massive multiplets" , respectively. The BPS bound [68] requires that

$$M_{ADM} \geq |Z|, \tag{1.13}$$

where M_{ADM} denotes the Arnowitt-Deser-Misner (ADM) mass [69]. At the event horizon, extremal BPS

[2]The non-BPS BH attractor Eqs. $\partial V_{BH} = 0$ with the condition $Z = 0$, $DZ \neq 0$ will be discussed later in the lectures.

[3]In case that the critical Hessian matrix has some "massless modes" (*i.e.* vanishing eigenvalues), one has to look at higher-order covariant derivatives of V_{BH} evaluated at the considered point, and study their sign. Depending on the configurations of the BH charges, one can obtain stable or unstable critical points.

Examples in literature of investigations beyond the Hessian level can be found in [10, 23, 24]. A detailed analysis of the stability of critical points of V_{BH} in (the large volume limit of) compactifications of Type IIA superstrings on CY_3s has been recently given in [34].

The issue of stability of non-BPS critical points of V_{BH} in homogeneous (not necessarily symmetric) $\mathcal{N} = 2$, $d = 4$ special Kähler geometries has been treated exhaustively in [36]. It was derived that all non-BPS critical points of V_{BH} in such geometries are stable, up to a certain number of "flat" directions (present at all order in the covariant differentiation of V_{BH}), which span a certain moduli space, pertaining to the considered class of solutions of the attractor Eqs.. The results of [36] hold in general for any theory (not necessarily involving supersymmetry) in which gravity is coupled to Abelian gauge vectors and with a scalar sigma model endowed with homogeneous geometry (see further below in the present lectures).

[4]For further elucidations, we refer the reader *e.g.* to the recent lectures of Sen [47], where important aspects of BH attractors are presented: the microscopic string theory counting of states explaining the macroscopic BH entropy and the treatment of higher-derivative terms in the actions.

BHs do saturate such a bound, whereas the non-BPS ones satisfy[5]

$$
\begin{array}{l}
\frac{1}{2}\text{-}BPS\text{:}\;\; 0<|Z|_H = M_{ADM,H};\\
non\text{-}BPS \left\{ \begin{array}{l} Z\neq 0\text{:}\;\; 0<|Z|_H < M_{ADM,H};\\ \\ Z=0\text{:}\;\; 0=|Z|_H < M_{ADM,H}, \end{array}\right.
\end{array}
\tag{1.14}
$$

where $M_{ADM,H}$ is obtained by extremizing $V_{BH}(\phi,Q)$ with respect to its dependence on the scalars:

$$
M_{ADM,H}(Q)=\sqrt{\left.V_{BH}(\phi,Q)\right|_{\partial_\phi V_{BH}=0}}. \tag{1.15}
$$

The (purely) charge-dependent BH entropy S_{BH} is given by the Bekenstein-Hawking entropy-area formula [67, 5]

$$
S_{BH}(Q)=\frac{A_H(Q)}{4}=\pi\left.V_{BH}(\phi,Q)\right|_{\partial_\phi V_{BH}=0}=\pi V_{BH}(\phi_H(Q),Q), \tag{1.16}
$$

where A_H is the area of the BH event horizon.

Non-degenerate, non-supersymmetric (non-BPS) extremal BH (and black string) attractors arise also in $\mathcal{N}=2$, $d=5$, 6 supergravity and in $\mathcal{N}>2$, $d=4$, 5, 6 extended supergravities (see *e.g.* [70, 71, 72, 73, 19, 74, 40, 38, 46, 49], and Refs. therein). In the present lectures we will focus on extremal BH attractors in $\mathcal{N}=2$, $d=4$ ungauged supergravity coupled to Abelian vector multiplets, where the scalar manifold parameterized by the scalars is endowed with the so-called special Kähler (SK) geometry (see Sect. 2).

Flux vacua (FV) became recently one of the new playgrounds for string theory, in general and in particular in the context of moduli stabilization (for an introduction to flux compactifications, see *e.g.* [75, 76, 77, 78, 79] and Refs. therein).

The advances of observational cosmology and the emergence of the so-called *"standard cosmological model"* enforce on string theory/supergravity a responsibility to address the current and future observations. This requires a solution of the problem of moduli stabilization. In the early Universe during inflation, all string theory moduli but the inflaton have to be stabilized, in order to produce an effective four-dimensional General Relativity and also in order for inflation to explain the cosmic microwave background observations. At the present time, all moduli have to be stabilized in a four-dimensional de Sitter space to explain dark energy and acceleration of the Universe which took place during the last few billion years.

The procedure of moduli stabilization in string theory consists of few steps.

One of the steps is the stabilization of moduli by fluxes in type IIB string theory, determining $d=4$ FV, with effective $\mathcal{N}=1$ local supersymmetry and complex structure moduli stabilized; an important feature of such a procedure is the non-stabilization of the Kähler moduli. However, the largest contribution to the counting of the Calabi-Yau vacua in the so-called *String Landscape* comes from the diversity of FV.

Thus, it is still interesting to study the mechanism of stabilization of the axion-dilaton and complex structure moduli in FV, ignoring the Kähler moduli. We will deal with such a scenario, in the particular

[5] Here and in what follows, the subscript "H" will denote values at the BH event horizon.

case in which the geometry of the complex structure moduli is SK, and not simply Kähler. In such a framework, it turns out that the Eqs. determining the FV configurations are closely related to the abovementioned extremal BH attractor Eqs. of $\mathcal{N}=2$, $d=4$ supergravity.

In the studies of FV one can start with an F-theory flux compactification on an elliptically fibered Calabi-Yau fourfold CY_4 in the orientifold limit in which $CY_4 = \frac{CY_3 \times T^2}{\mathbb{Z}_2}$, where T^2 is the two-torus. In type IIB string theory, this is equivalent to compactifying on the orientifold limit of CY_3. The resulting low energy, $d=4$ effective theory is $\mathcal{N}=1$ supergravity, where the information on string theory choice of compactification is encoded into a flux superpotential W and a Kähler potential K. As explained above, we assume that both flux superpotential W and Kähler potential K depend only on the complex structure (CS) moduli of CY_4, spanning the CS moduli space M. Because of the orientifold limit of CY_4, M has the product structure $\mathcal{M} = \mathcal{M}_{CS}(CY_3) \times \mathcal{M}_\tau$ (see Eq. (4.1.1) further below), where $\mathcal{M}_{CS}(CY_3)$ (simply named $\mathcal{M}_{CS}$ further below) is the CS moduli space of CY_3 and $\mathcal{M}_\tau$ is the moduli space of the elliptic curve T^2 spanned by the axion-dilaton τ (named t^0 in the treatment of the present lectures). Let us here just mention that in order to stabilize also the Kähler moduli of CY_4, one should incorporate the non-perturbative string effects (see *e.g.* [80]), which however we will not discuss here.

The potential in the effective $\mathcal{N}=1$, $d=4$ supergravity theory, in the Planckian units set equal to one, is given by [81, 82]

$$V_{\mathcal{N}=1} = e^K \left(\sum_{A=0}^{h_{2,1}(CY_3)} |D_A W|^2 - 3|W|^2 \right) = \sum_{A=0}^{h_{2,1}(CY_3)} |D_A Z|^2 - 3|Z|^2 \ , \tag{1.17}$$

where $A=0$ refers to the axion-dilaton $\tau \equiv t^0$ and $A = i \in \{1, \cdots, h_{2,1}(CY_3)\}$ to the CS moduli t^i of CY_3 ($h_{2,1} \equiv dim\left(H^{2,1}(CY_3)\right)$; see Subsect. 4.1). We defined $Z \equiv e^{\frac{K}{2}} W$, as for the extremal BH attractors in $\mathcal{N}=2$, $d=4$ supergravity, even if the analogy is only formal, because in the present $d=4$ framework with $\mathcal{N}=1$ local supersymmetry there is no central charge at all.

The real Kähler potential of the effective $\mathcal{N}=1$, $d=4$ supergravity theory reads (see Eq. (4.1.2.18) below)

$$K = -ln\langle \Omega_4, \bar{\Omega}_4 \rangle = -ln(\langle \Omega_1, \bar{\Omega}_1 \rangle) - ln(\langle \Omega_3, \bar{\Omega}_3 \rangle) \ , \tag{1.18}$$

where Ω_4 is a nowhere vanishing holomorphic 4-form defined on CY_4. In the orientifold limit, Ω_4 is a product of an appropriate holomorphic 3-form Ω_3 of CY_3 and the holomorphic 1-form Ω_1 of the torus T^2 (see Subsubsect. 4.1.2).

The flux holomorphic superpotential W is defined as a section of the line bundle $\mathcal{L}$ by [83, 84] (see Eq. (4.1.2.19) below)

$$W \equiv Z e^{-\frac{K}{2}} = \langle \mathcal{F}_4, \Omega_4 \rangle \equiv \int_{CY_4} \mathcal{F}_4 \wedge \Omega_4 \ , \tag{1.19}$$

where $\mathcal{F}_4 \in H^4(CY_4)$ is the 4-form flux.

In generic local "flat" coordinates of M (with 0 and A-indices respectively referring to the axion-dilaton and CS moduli of CY_3), $\mathcal{F}_4$ enjoys the following Hodge decomposition, which we present here in terms of Z for the sake of comparison with its "BH-counterpart" (*i.e.* the Hodge decomposition (1.5)

of the 3-form flux $\mathcal{H}_3$) ($\hat{\Omega}_4 \equiv e^{\frac{1}{2}K}\Omega_4$; see definition (4.1.2.21) and Eq. (4.1.3.6) below):

$$\mathfrak{F}_4 = 2Re\left[\overline{Z}\hat{\Omega}_4 - \delta^{A\overline{B}}\left(\overline{D}_{\overline{B}}\overline{Z}\right)D_A\hat{\Omega}_4 + \delta^{A\overline{B}}\left(\overline{D}_{\underline{\overline{0}}}\overline{D}_{\overline{B}}\overline{Z}\right)D_{\underline{0}}D_A\hat{\Omega}_4\right]. \tag{1.20}$$

By imposing the supersymmetry-preserving condition $DZ = 0$ (formally identical to the one appearing in the abovementioned theory of extremal BH attractors in $\mathcal{N} = 2$, $d = 4$ supergravity), the identity (1.20) becomes a supersymmetric FV Attractor Eq.. Indeed, the left-hand side (l.h.s.) depends on fluxes and the r.h.s. depends on axion-dilaton and on CS moduli of CY_3; thus, the solution stabilizes the axion-dilaton and the CS moduli of CY_3 purely in terms of fluxes: beside Eq. (1.7), one gets

$$\mathfrak{F}_4 = 2Re\left[\overline{Z}\hat{\Omega}_4 + \delta^{A\overline{B}}\left(\overline{D}_{\underline{\overline{0}}}\overline{D}_{\overline{B}}\overline{Z}\right)D_{\underline{0}}D_A\hat{\Omega}_4\right]_{DZ=0}. \tag{1.21}$$

Let us now compare the supersymmetric FV Attractor Eqs. (1.21) with their "BH-counterpart", *i.e.* with the BPS extremal BH Attractor Eqs. (1.7).

In the case of FV, instead of the imaginary part we have a real part of a somewhat analogous expression: this is due to the fact that for FV one has a 4-form flux $\mathfrak{F}_4$ on a CY_3 orientifold rather than of a 3-form flux $\mathcal{H}_3$ on CY_3.

The other significant difference is in the second term in the r.h.s. of Eq. (1.21). This term, absent in the BH case, is proportional to the second-order covariant derivative of Z along the τ direction and one of the directions pertaining to the CS moduli of CY_3. In the BH case there is a relation $D_iD_jZ = iC_{ijk}\overline{D}^k\overline{Z}$ (see the second of Eqs. (2.17) below), and therefore the second covariant derivative of Z is not an independent term for BH, differently from $D_{\underline{0}}D_IZ$ in the FV case, which is an independent term in the decomposition of forms.

The absence of such a term in the BPS extremal BH Attractor Eqs. (1.7) does not allow for non-degenerate BPS extremal BH attractors with vanishing central charge: indeed, on the BH side $Z = 0$ and $DZ = 0$ yield $V_{BH} = 0$. This limit case corresponds to a classical *"small"* extremal BH, exhibiting a naked singularity because the area of the BH event horizon vanishes. The Attractor Mechanism in such a case simply ceases to hold, because for $Z = 0$ and $DZ = 0$ the BPS extremal BH Attractor Eqs. (1.3) admit as unique solution $Q = 0$.

The same does not happen on the FV side. Indeed, by substituting $Z = 0$ and $DZ = 0$ into the Hodge decomposition (1.20) does not generate any unconsistency: the second term in the r.h.s. of Eq. (1.21) provides a consistent solution for superymmetric Minkowski vacua (with $DZ = 0$ and $V_{\mathcal{N}=1} = 0$). Of course, more general supersymmetric solutions with $Z \neq 0$ are allowed, and they correspond to supersymmetric AdS FV (see *e.g.* [11, 85]).

The aim of the present paper is to show the Attractor Mechanism at work in two completely different $d = 4$ frameworks: extremal BH in $\mathcal{N} = 2$ supergravity and $\mathcal{N} = 1$ flux compactifications.

The plan of the paper is as follows.

In Sect. 2 we recall the fundamentals of the special Kähler geometry, underlying the vector multiplets' scalar manifold of $\mathcal{N} = 2$, $d = 4$ ungauged supergravity, as well as the complex structure moduli space of certain $\mathcal{N} = 1$, $d = 4$ supergravities obtained by consistently orientifolding of $\mathcal{N} = 2$ theories, such as Type IIB compactified on $\frac{CY_3 \times T^2}{\mathbb{Z}_2}$.

Sect. 3 gives an introduction to the issue of the Attractor Mechanism in the framework where it was originally discovered by Ferrara, Kallosh and Strominger [1]-[5], namely in the stabilization of the vector multiplet' scalars near the event horizon of an extremal, static, spherically symmetric and asymptotically flat BH in $\mathcal{N}=2$, $d=4$ ungauged supergravity.

Subsect. 3.1 presents the so-called *"criticality conditions"* approach to the Attractor Mechanism, in which the purely charge-dependent stabilized configurations of the scalars at the BH horizon can be computed as the critical points of a certain real positive BH effective potential V_{BH}, whose classification is given in Subsubsect. 3.1.1. The stability of the critical points of V_{BH} is then analyzed in Subsect. 3.2, both in the general case of n_V moduli (Subsubsect. 3.2.1) and in the 1-modulus case (Subsubsect. 3.2.2).

Subsect. 3.3 presents another, equivalent approach to the Attractor Mechanism, recently named *"New Attractor"* approach. In Subsubsect. 3.3.2 it is exploited in a general $\mathcal{N}=2$, $d=4$ supergravity framework, by substituting the (various classes of) criticality conditions of V_{BH} into some geometrical identities of special Kähler geometry, expressing nothing but a change of basis between *"dressed"* and *"undressed"* charges, and derived in Subsubsect. 3.3.1.

Subsect. 3.4 implements the *"New Attractor"* approach in a stringy framework, namely in Type IIB compactified on CY_3. In Subsubsect. 3.4.2 the (various classes of) criticality conditions of V_{BH} are inserted into some general identities (equivalent to the identities derived in Subsubsect. 3.3.1), expressing the decomposition of the real, Kähler gauge-invariant 3-form flux $\mathcal{H}_3$ along the third Dalbeault cohomogy of CY_3, and derived in Subsubsect. 3.4.1.

Sect. 4 deals with the Attractor Mechanism in a completely different framework, namely in $\mathcal{N}=1$, $d=4$ ungauged supergravity obtained by consistently orientifolding the $\mathcal{N}=2$ theory, and thus mantaining a special Kähler geometry of the manifold of the scalars surviving the orientifolding. In the considered example of Type IIB associated with $\frac{CY_3\times T^2}{\mathbb{Z}_2}$, the Attractor Mechanism determines the stabilization of the universal axion-dilaton and of the complex structure moduli in terms of the Ramond-Ramond (RR) and Neveu-Schwarz-Neveu-Schwarz (NSNS) fluxes.

Subsect. 4.1 introduces the fundamentals of the geometry of (the moduli space of) CY_3 orientifolds: the vielbein and the metric tensor (Subsubsect. 4.1.1), the relevant 1-, 3- and 4-forms (Subsubsect. 4.1.2), and the Hodge decomposition of the real, Kähler gauge-invariant 4-form flux $\mathfrak{F}_4$, unifying the RR 3-form flux $\mathfrak{F}_3$ with the NSNS 3-form flux $\mathfrak{H}_3$ (Subsubsect. 4.1.3).

Subsect. 4.2 presents the so-called *"criticality conditions"* approach to the Attractor Mechanism in flux vacua (FV) compactifications of the kind considered above, in which the (complex structure) moduli space is endowed with special Kähler geometry. The purely flux-dependent stabilized vacuum configurations of the axion-dilaton and complex structure moduli can be computed as the critical points of a certain real (not necessarily positive) FV effective potential $V_{\mathcal{N}=1}$. Since (differently from its BH $\mathcal{N}=2$ counterpart V_{BH}) $V_{\mathcal{N}=1}$ has no definite sign, the FV attractor configurations can correspond to a de Sitter (dS, $V_{\mathcal{N}=1}>0$), Minkowski ($V_{\mathcal{N}=1}=0$) or anti-de Sitter (AdS, $V_{\mathcal{N}=1}<0$) vacuum.

Finally, Subsect. 4.3 implements the *"New Attractor"* approach to the Attractor mechanism in the considered class of FV compactifications, in the case of supersymmetric vacuum configurations.

The supersymmetric criticality conditions of $V_{\mathcal{N}=1}$ are inserted into the Hodge decomposition of the 4-form flux $\mathfrak{F}_4$, and the resulting supersymmetric FV Attractor Eqs. lead to the classification of the supersymmetric FV into three general families.

Two Appendices, containing technical details, conclude the lectures.

2 Special Kähler Geometry

In the present Section we briefly recall the fundamentals of the SK geometry underlying the scalar manifold $\mathcal{M}_{n_V}$ of $\mathcal{N}=2$, $d=4$ supergravity, n_V being the number of Abelian vector supermultiplets coupled to the supergravity multiplet ($dim_{\mathbb{C}}\mathcal{M}_{n_V}=n_V$) (see *e.g.* [86, 87, 88, 89, 90, 91, 92, 93, 63, 64]).

It is convenient to switch from the Riemannian $2n_V$-dim. parameterization of $\mathcal{M}_{n_V}$ given by the local real coordinates $\{\phi^a\}_{a=1,\dots,2n_V}$ to the Kähler n_V-dim. holomorphic/antiholomorphic parameterization given by the local complex coordinates $\left\{z^i,\overline{z}^{\overline{i}}\right\}_{i,\overline{i}=1,\dots,n_V}$. This corresponds to the following *unitary Cayley transformation*:

$$z^k\equiv\frac{\varphi^{2k-1}+i\varphi^{2k}}{\sqrt{2}},\quad k=1,\dots,n_V. \tag{2.1}$$

The metric structure of $\mathcal{M}_{n_V}$ is given by the covariant (special) Kähler metric tensor $g_{i\overline{j}}\left(z,\overline{z}\right)=\partial_i\overline{\partial}_{\overline{j}}K\left(z,\overline{z}\right)$, $K\left(z,\overline{z}\right)$ being the real Kähler potential.

Usually, the $n_V\times n_V$ Hermitian matrix $g_{i\overline{j}}$ is assumed to be non-degenerate (*i.e.* invertible, with non-vanishing determinant and rank n_V) and with strict positive Euclidean signature (*i.e.* with all strictly positive eigenvalues) *globally* in $\mathcal{M}_{n_V}$. We will so assume, even though we will be concerned mainly with the properties of $g_{i\overline{j}}$ at those peculiar points of $\mathcal{M}_{n_V}$ which are critical points of V_{BH}.

It is worth here remarking that various possibilities arise when going beyond the assumption of *global strict regular* $g_{i\overline{j}}$, namely:

- (*locally*) *not strictly regular* $g_{i\overline{j}}$, *i.e.* a (*locally*) non-invertible metric tensor, with some strictly positive and some vanishing eigenvalues (rank $<n_V$);

- (*locally*) *non-regular non-degenerate* $g_{i\overline{j}}$, *i.e.* a (*locally*) invertible metric tensor with *pseudo-Euclidean signature*, namely with some strictly positive and some strictly negative eigenvalues (rank $=n_V$);

- (*locally*) *non-regular degenerate* $g_{i\overline{j}}$, *i.e.* a (*locally*) non-invertible metric tensor with some strictly positive, some strictly negative, and some vanishing eigenvalues (rank $<n_V$).

The *local* violation of strict regularity of $g_{i\overline{j}}$ would produce some kind of "phase transition" in the SKG endowing $\mathcal{M}_{n_V}$, corresponding to a breakdown of the 1-dim. effective Lagrangian picture (see [5], [94], and also [18] and [74]) of $d=4$ (extremal) BHs obtained by integrating all massive states of the theory out, unless new massless states appear [5].

The previously mentioned $\mathcal{N}=2$, $d=4$ covariantly holomorphic *central charge function* is defined as

$$\begin{aligned}&Z\left(z,\overline{z};q,p\right)\equiv Q\epsilon V\left(z,\overline{z}\right)=q_\Lambda L^\Lambda\left(z,\overline{z}\right)-p^\Lambda M_\Lambda\left(z,\overline{z}\right)=e^{\frac{1}{2}K\left(z,\overline{z}\right)}Q\epsilon\Pi\left(z\right)=\\&=e^{\frac{1}{2}K\left(z,\overline{z}\right)}\left[q_\Lambda X^\Lambda\left(z\right)-p^\Lambda F_\Lambda\left(z\right)\right]\equiv e^{\frac{1}{2}K\left(z,\overline{z}\right)}W\left(z;q,p\right),\end{aligned} \tag{2.2}$$

where ϵ is the $(2n_V+2)$-dim. square symplectic metric (subscripts denote dimensions of square sub-blocks)

$$\epsilon \equiv \left(\begin{array}{cc} 0_{n_V+1} & -\mathbb{I}_{n_V+1} \\ \\ \mathbb{I}_{n_V+1} & 0_{n_V+1} \end{array} \right), \tag{2.3}$$

and $V(z,\overline{z})$ and $\Pi(z)$ respectively stand for the $(2n_V+2)\times 1$ covariantly holomorphic (Kähler weights $(1,-1)$) and holomorphic (Kähler weights $(2,0)$) period vectors in symplectic basis:

$$\overline{D}_{\overline{i}}V(z,\overline{z}) = \left(\overline{\partial}_{\overline{i}} - \tfrac{1}{2}\overline{\partial}_{\overline{i}}K\right)V(z,\overline{z}) = 0, \quad D_iV(z,\overline{z}) = \left(\partial_i + \tfrac{1}{2}\partial_i K\right)V(z,\overline{z})$$

$$\Updownarrow$$

$$V(z,\overline{z}) = e^{\frac{1}{2}K(z,\overline{z})}\Pi(z), \quad \overline{D}_{\overline{i}}\Pi(z) = \overline{\partial}_{\overline{i}}\Pi(z) = 0, \quad D_i\Pi(z) = (\partial_i + \partial_i K)\Pi(z), \tag{2.4}$$

$$\Pi(z) \equiv \left(\begin{array}{c} X^\Lambda(z) \\ \\ F_\Lambda(X(z)) \end{array} \right) = exp\left(-\tfrac{1}{2}K(z,\overline{z})\right) \left(\begin{array}{c} L^\Lambda(z,\overline{z}) \\ \\ M_\Lambda(z,\overline{z}) \end{array} \right),$$

with $X^\Lambda(z)$ and $F_\Lambda(X(z))$ being the holomorphic sections of the $U(1)$ line (Hodge) bundle over $\mathcal{M}_{n_V}$. $W(z;q,p)$ is the so-called *holomorphic* $\mathcal{N}=2$, $d=4$ *central charge function*, also named $\mathcal{N}=2$ *superpotential*.

Up to some particular choices of local symplectic coordinates in $\mathcal{M}_{n_V}$, the covariant symplectic holomorphic sections $F_\Lambda(X(z))$ may be seen as derivatives of an *holomorphic prepotential* function F (with Kähler weights $(4,0)$):

$$F_\Lambda(X(z)) = \frac{\partial F(X(z))}{\partial X^\Lambda}. \tag{2.5}$$

In $\mathcal{N}=2$, $d=4$ supergravity the holomorphic function F is constrained to be homogeneous of degree 2 in the contravariant symplectic holomorphic sections $X^\Lambda(z)$, *i.e.* (see [64] and Refs. therein)

$$2F(X(z)) = X^\Lambda(z)F_\Lambda(X(z)). \tag{2.6}$$

The normalization of the holomorphic period vector $\Pi(z)$ is such that

$$K(z,\overline{z}) = -ln\left[i\left\langle\Pi(z),\overline{\Pi}(\overline{z})\right\rangle\right] \equiv -ln\left[i\Pi^T(z)\epsilon\overline{\Pi}(\overline{z})\right] = -ln\left\{i\left[\overline{X}^\Lambda(\overline{z})F_\Lambda(z) - X^\Lambda(z)\overline{F}_\Lambda(\overline{z})\right]\right\}, \tag{2.7}$$

where $\langle\cdot,\cdot\rangle$ stands for the symplectic scalar product defined by ϵ.

Note that under a Kähler transformation

$$K(z,\overline{z}) \longrightarrow K(z,\overline{z}) + f(z) + \overline{f}(\overline{z}) \tag{2.8}$$

($f(z)$ being a generic holomorphic function), the holomorphic period vector transforms as

$$\Pi(z) \longrightarrow \Pi(z)e^{-f(z)} \Leftrightarrow X^\Lambda(z) \longrightarrow X^\Lambda(z)e^{-f(z)}. \tag{2.9}$$

This means that, at least locally, the contravariant holomorphic symplectic sections $X^\Lambda(z)$ can be regarded as a set of homogeneous coordinates on $\mathcal{M}_{n_V}$, provided that the Jacobian complex $n_V \times n_V$ holomorphic matrix

$$e_i^a(z) \equiv \frac{\partial}{\partial z^i}\left(\frac{X^a(z)}{X^0(z)}\right), \quad a = 1,...,n_V \tag{2.10}$$

is invertible. If this is the case, then one can introduce the local projective symplectic coordinates

$$t^a(z) \equiv \frac{X^a(z)}{X^0(z)}, \tag{2.11}$$

and the SKG of $\mathcal{M}_{n_V}$ turns out to be based on the holomorphic prepotential $\mathcal{F}(t) \equiv \left(X^0\right)^{-2} F(X)$. By using the t-coordinates, Eq. (2.7) can be rewritten as follows ($\mathcal{F}_a(t) = \partial_a \mathcal{F}(t)$, $\bar{t}^a = \overline{t^a}$, $\overline{\mathcal{F}}_a\left(\bar{t}\right) = \overline{\mathcal{F}_a(t)}$):

$$K\left(t,\bar{t}\right) = -ln\left\{i\left|X^0(z(t))\right|^2\left[2\left(\mathcal{F}(t) - \overline{\mathcal{F}}\left(\bar{t}\right)\right) - \left(t^a - \bar{t}^a\right)\left(\mathcal{F}_a(t) + \overline{\mathcal{F}}_a\left(\bar{t}\right)\right)\right]\right\}. \tag{2.12}$$

By performing a Kähler gauge-fixing with $f(z) = ln\left(X^0(z)\right)$, yielding that $X^0(z) \longrightarrow 1$, one thus gets

$$K\left(t,\bar{t}\right)\big|_{X^0(z)\longrightarrow 1} = -ln\left\{i\left[2\left(\mathcal{F}(t) - \overline{\mathcal{F}}\left(\bar{t}\right)\right) - \left(t^a - \bar{t}^a\right)\left(\mathcal{F}_a(t) + \overline{\mathcal{F}}_a\left(\bar{t}\right)\right)\right]\right\}. \tag{2.13}$$

In particular, one can choose the so-called *special coordinates*, i.e. the system of local projective t-coordinates such that

$$e_i^a(z) = \delta_i^a \Leftrightarrow t^a(z) = z^i\left(+c^i,\ c^i \in \mathbb{C}\right). \tag{2.14}$$

Thus, Eq. (2.13) acquires the form

$$K\left(t,\bar{t}\right)\big|_{X^0(z)\longrightarrow 1, e_i^a(z)=\delta_i^a} = -ln\left\{i\left[2\left(\mathcal{F}(z) - \overline{\mathcal{F}}(\bar{z})\right) - \left(z^j - \bar{z}^{\bar{j}}\right)\left(\mathcal{F}_j(z) + \overline{\mathcal{F}}_{\bar{j}}(\bar{z})\right)\right]\right\}. \tag{2.15}$$

Moreover, it should be recalled that Z has Kähler weights $(p,\bar{p}) = (1,-1)$, and therefore its Kähler-covariant derivatives read

$$D_i Z = \left(\partial_i + \frac{1}{2}\partial_i K\right) Z, \quad \overline{D}_{\bar{i}} Z = \left(\overline{\partial}_{\bar{i}} - \frac{1}{2}\overline{\partial}_{\bar{i}} K\right) Z. \tag{2.16}$$

The fundamental differential relations of SK geometry are[6] (see *e.g.* [64]):

$$\begin{cases} D_i Z = Z_i; \\ D_i Z_j = i C_{ijk} g^{k\bar{k}} \overline{D}_{\bar{k}} \overline{Z} = i C_{ijk} g^{k\bar{k}} \overline{Z}_{\bar{k}}; \\ D_i \overline{D}_{\bar{j}} \overline{Z} = D_i \overline{Z}_{\bar{j}} = g_{i\bar{j}} \overline{Z}; \\ D_i \overline{Z} = 0, \end{cases} \tag{2.17}$$

where the first relation is nothing but the definition of the so-called *matter charges* Z_i, and the fourth relation expresses the Kähler-covariant holomorphicity of Z. C_{ijk} is the rank-3, completely symmetric,

[6] Actually, there are different (equivalent) defining approaches to SK geometry. For subtleties and further elucidation concerning such an issue, see *e.g.* [95] and [96].

covariantly holomorphic tensor of SK geometry (with Kähler weights $(2,-2)$) (see *e.g.*[7] [64, 91, 92]):

$$
\left\{
\begin{array}{l}
C_{ijk} = \left\langle D_i D_j V, D_k V\right\rangle = e^K \left(\partial_i \mathcal{N}_{\Lambda\Sigma}\right) D_j X^\Lambda D_k X^\Sigma = \\
= e^K \left(\partial_i X^\Lambda\right)\left(\partial_j X^\Sigma\right)\left(\partial_k X^\Xi\right)\partial_\Xi \partial_\Sigma F_\Lambda\left(X\right) \equiv e^K W_{ijk}, \quad \overline{\partial}_{\overline{l}} W_{ijk} = 0; \\
C_{ijk} = D_i D_j D_k \mathcal{S}, \quad \mathcal{S} \equiv -i L^\Lambda L^\Sigma Im\left(F_{\Lambda\Sigma}\right), \quad F_{\Lambda\Sigma} \equiv \frac{\partial F_\Lambda}{\partial X^\Sigma}, F_{\Lambda\Sigma} \equiv F_{(\Lambda\Sigma)} \ ; \\
C_{ijk} = -i g_{i\overline{l}} \overline{f}^{\overline{l}}_\Lambda D_j D_k L^\Lambda, \quad \overline{f}^{\overline{l}}_\Lambda \left(\overline{DL}^\Lambda_{\overline{s}}\right) \equiv \delta^{\overline{l}}_{\overline{s}};
\end{array}
\right. \tag{2.18}
$$

$$\overline{D}_{\overline{i}} C_{jkl} = 0 \ (\textit{covariant holomorphicity});$$

$$R_{i\overline{j}k\overline{l}} = -g_{i\overline{j}} g_{k\overline{l}} - g_{i\overline{l}} g_{k\overline{j}} + C_{ikp}\overline{C}_{\overline{j}\overline{l}\overline{p}} g^{p\overline{p}} \ \text{(usually named } \textit{SKG constraints}\text{)};$$

$$D_{[i} C_{j]kl} = 0,$$

where the last property is a consequence, through the SKG constraints and the covariant holomorphicity of C_{ijk}, of the Bianchi identities for the Riemann tensor $R_{i\overline{j}k\overline{l}}$ (see *e.g.* [91]), and square brackets denote antisymmetrization with respect to enclosed indices. For later convenience, it is here worth writing the expression for the holomorphic covariant derivative of C_{ijk}:

$$D_i C_{jkl} = D_{(i} C_{j)kl} = \partial_i C_{jkl} + \left(\partial_i K\right) C_{jkl} + \Gamma_{ij}{}^m C_{mkl} + \Gamma_{ik}{}^m C_{mjl} + \Gamma_{il}{}^m C_{mjk}. \tag{2.19}$$

It is worth recalling that in a generic Kähler geometry $R_{i\overline{j}k\overline{l}}$ reads

$$
\begin{array}{l}
R_{i\overline{j}k\overline{l}} = g^{m\overline{n}}\left(\overline{\partial}_{\overline{l}}\overline{\partial}_{\overline{j}}\partial_m K\right)\partial_i \overline{\partial}_{\overline{n}} \partial_k K - \overline{\partial}_{\overline{l}}\partial_i \overline{\partial}_{\overline{j}} \partial_k K = g_{k\overline{n}} \partial_i \overline{\Gamma}_{\overline{l}\overline{j}}{}^{\overline{n}} = g_{n\overline{l}} \overline{\partial}_{\overline{j}} \Gamma_{ki}{}^n, \\
\overline{R_{i\overline{j}k\overline{l}}} = R_{j\overline{i}l\overline{k}} \qquad (\textit{reality}), \\
\Gamma_{ij}{}^l = -g^{l\overline{l}} \partial_i g_{j\overline{l}} = -g^{l\overline{l}} \partial_i \overline{\partial}_{\overline{l}} \partial_j K = \Gamma_{(ij)}{}^l,
\end{array} \tag{2.20}
$$

where $\Gamma_{ij}{}^l$ stand for the Christoffel symbols of the second kind of the Kähler metric $g_{i\overline{j}}$.

In the first of Eqs. (2.18), a fundamental entity, the so-called kinetic matrix $\mathcal{N}_{\Lambda\Sigma}\left(z,\overline{z}\right)$ of $\mathcal{N}=2$, $d=4$ supergravity, has been introduced (see also Eq. (3.4.1.9) further below). It is an $(n_V+1)\times(n_V+1)$ complex symmetric, moduli-dependent, Kähler gauge-invariant matrix defined by the following fundamental *Ansätze* of SKG, solving the *SKG constraints* (given by the third of Eqs. (2.18)):

$$M_\Lambda = \mathcal{N}_{\Lambda\Sigma} L^\Sigma, \quad D_i M_\Lambda = \overline{\mathcal{N}}_{\Lambda\Sigma} D_i L^\Sigma. \tag{2.21}$$

By introducing the $(n_V+1)\times(n_V+1)$ complex matrices $(I = 1, ..., n_V+1)$

$$f_I^\Lambda\left(z,\overline{z}\right) \equiv \left(\overline{D}_{\overline{i}}\overline{L}^\Lambda\left(z,\overline{z}\right), L^\Lambda\left(z,\overline{z}\right)\right), \quad h_{I\Lambda}\left(z,\overline{z}\right) \equiv \left(\overline{D}_{\overline{i}}\overline{M}_\Lambda\left(z,\overline{z}\right), M_\Lambda\left(z,\overline{z}\right)\right), \tag{2.22}$$

the *Ansätze* (2.21) univoquely determine $\mathcal{N}_{\Lambda\Sigma}\left(z,\overline{z}\right)$ as

$$\mathcal{N}_{\Lambda\Sigma}\left(z,\overline{z}\right) = h_{I\Lambda}\left(z,\overline{z}\right)\circ\left(f^{-1}\right)^I_\Sigma\left(z,\overline{z}\right), \tag{2.23}$$

[7] Notice that the third of Eqs. (2.18) correctly defines the Riemann tensor $R_{i\overline{j}k\overline{l}}$, and it is actual the opposite of the one which may be found in a large part of existing literature. Such a formulation of the so-called *SKG constraints* is well defined, because, as we will mention at the end of Sect. 3.2, it yields negative values of the constant scalar curvature of ($n_V=$ 1-dim.) homogeneous symmetric compact SK manifolds.

where $\circ$ denotes the usual matrix product, and $\left(f^{-1}\right)_{\Sigma}^{I} f_{I}^{\Lambda}=\delta_{\Sigma}^{\Lambda}$, $\left(f^{-1}\right)_{\Lambda}^{I} f_{J}^{\Lambda}=\delta_{J}^{I}$.

The covariantly holomorphic $(2n_V+2)\times 1$ period vector $V(z,\overline{z})$ is *symplectically orthogonal* to all its Kähler-covariant derivatives:

$$\left\{\begin{array}{l}\langle V(z,\overline{z}), D_i V(z,\overline{z})\rangle=0; \\ \left\langle V(z,\overline{z}), \overline{D}_{\overline{i}} V(z,\overline{z})\right\rangle=0; \\ \left\langle V(z,\overline{z}), D_i \overline{V}(z,\overline{z})\right\rangle=0; \\ \left\langle V(z,\overline{z}), \overline{D}_{\overline{i}} \overline{V}(z,\overline{z})\right\rangle=0.\end{array}\right. \tag{2.24}$$

Morover, it holds that

$$\begin{aligned} & g_{i\overline{j}}(z,\overline{z})=-i\left\langle D_i V(z,\overline{z}), \overline{D}_{\overline{j}} \overline{V}(z,\overline{z})\right\rangle= \\ & =-2Im\left(\mathcal{N}_{\Lambda\Sigma}(z,\overline{z})\right) D_i L^{\Lambda}(z,\overline{z}) \overline{D}_{\overline{i}} \overline{L}^{\Sigma}(z,\overline{z})=2Im\left(F_{\Lambda\Sigma}(z)\right) D_i L^{\Lambda}(z,\overline{z}) \overline{D}_{\overline{i}} \overline{L}^{\Sigma}(z,\overline{z}); \end{aligned} \tag{2.25}$$

$$\left\langle V(z,\overline{z}), D_i \overline{D}_{\overline{j}} V(z,\overline{z})\right\rangle=iC_{ijk} g^{k\overline{k}}\left\langle V(z,\overline{z}), \overline{D}_{\overline{k}} \overline{V}(z,\overline{z})\right\rangle=0. \tag{2.26}$$

3 Extremal Black Hole Attractor Equations in $\mathcal{N}=2$, $d=4$ (ungauged) Supergravity

3.1 Black Hole Effective Potential and "*Criticality Conditions*" *Approach*

In $\mathcal{N}=2$, $d=4$ supergravity the "effective BH potential" reads [3, 4, 64]

$$V_{BH}(z,\overline{z};q,p)=|Z|^2(z,\overline{z};q,p)+g^{i\overline{j}}(z,\overline{z}) D_i Z(z,\overline{z};q,p) \overline{D}_{\overline{j}} \overline{Z}(z,\overline{z};q,p)=I_1(z,\overline{z};q,p)\geqslant 0, \tag{3.1.1}$$

where I_1 is the first, positive-definite real invariant I_1 of SK geometry (see *e.g.* [23, 64]). It should be noticed that V_{BH} can also be obtained by left-multiplying the SKG vector identity (3.3.1.16) by the $1\times(2n_V+2)$ complex moduli-dependent vector $-\frac{1}{2}Q\mathcal{M}(\mathcal{N})$; indeed, since the matrix $\mathcal{M}(\mathcal{N})$ is symplectic, one finally gets [3, 4, 64]

$$V_{BH}(z,\overline{z};q,p)=-\frac{1}{2}Q\mathcal{M}(\mathcal{N})Q^T. \tag{3.1.2}$$

It is interesting to remark that the result (3.1.2) can be elegantly obtained from the SK geometry identities (3.3.1.16) by making use of the following relation (see [19], where a generalization for $\mathcal{N}>2$-extended supergravities is also given):

$$\frac{1}{2}\left(\mathcal{M}(\mathcal{N})+i\Omega\right)\mathcal{V}=i\Omega\mathcal{V}\Leftrightarrow\mathcal{M}(\mathcal{N})\mathcal{V}=i\Omega\mathcal{V}, \tag{3.1.3}$$

where $\mathcal{V}$ is a $(2n_V+2)\times(n_V+1)$ matrix defined as follows:

$$\mathcal{V}\equiv\left(V,\overline{D}_{\overline{1}}\overline{V},...,\overline{D}_{\overline{n_V}}\overline{V}\right). \tag{3.1.4}$$

By differentiating Eq. (3.1.1) with respect to the moduli, the criticality conditions of V_{BH} can be easily shown to acquire the form [5]

$$D_i V_{BH}=\partial_i V_{BH}=0\Leftrightarrow 2\overline{Z}D_i Z+g^{j\overline{j}}\left(D_i D_j Z\right)\overline{D}_{\overline{j}}\overline{Z}=0. \tag{3.1.5}$$

These are the what one should rigorously refer to as the $\mathcal{N}=2$, $d=4$ supergravity Attractor Eqs. (AEs).

In the present work, we will call *AEs* also some *geometrical identities* evaluated along the criticality conditions of the relevant "effective potential". Indeed, both for extremal BHs attractors in $\mathcal{N}=2$, $d=4$ supergravity and for FV attractors in $\mathcal{N}=1$, $d=4$ supergravity (*at least* for the one coming from some peculiar compactifications of superstrings: see Sect. 4), there exist two different approaches to determining the attractors:

i) the so-called *"criticality conditions" approach*, based on the direct solution of the conditions giving the stationary points of the relevant "effective potential";

ii) the so-called *"new attractor" approach*, based on the solution of some fundamental geometrical identities evaluated along the criticality conditions of the relevant "effective potential".

Such two approaches are completely equivalent. Dependingly on the considered frameworks, it can be convenient to exploit one approach rather than the other one (see *e.g.* [26] for an explicit case).

By using the relations (2.17), the $\mathcal{N}=2$ AEs (3.1.5) can be recast as follows [5]:

$$D_iV_{BH}=\partial_iV_{BH}=2\overline{Z}D_iZ+iC_{ijk}g^{j\bar{l}}g^{k\overline{m}}\left(\overline{D}_{\bar{l}}\overline{Z}\right)\overline{D}_{\overline{m}}\overline{Z}. \tag{3.1.6}$$

Eqs. (3.1.6) are nothing but the relations between the $\mathcal{N}=2$ central charge function Z (*graviphoton charge*) and the n_V *matter charges* Z_i (coming from the n_V Abelian vector supermultiplets), holding at the critical points of V_{BH} in the SK scalar manifold $\mathcal{M}_{n_V}$. As it is seen, the tensor C_{ijk} plays a key role.

It is known that static, spherically symmetric, asymptotically flat extremal BHs in $d=4$ are described by an effective $d=1$ Lagrangian ([5], [94], and also [18] and [74]), with an effective scalar potential and effective fermionic "mass terms" terms controlled by the field-strength fluxes vector Q defined by Eq. (1.1). The *"apparent" gravitino mass* is given by Z, whereas the $n_V\times n_V$ *gaugino mass matrix* Λ_{ij} reads (see the second Ref. of [92])

$$\Lambda_{ij}=-iD_iZ_j=C_{ijk}g^{k\bar{k}}\overline{Z}_{\bar{k}}=\Lambda_{(ij)}. \tag{3.1.7}$$

Note that Λ_{ij} is part of the holomorphic/anti-holomorphic form of the $2n_V\times 2n_V$ covariant Hessian of Z, which is nothing but the holomorphic/anti-holomorphic form of the scalar mass matrix. The *supersymmetry order parameters*, related to the mixed gravitino-gaugino couplings, are given by the *matter charge(function)s* $D_iZ=Z_i$ (see the first of Eqs. (2.17)).

By assuming that the Kähler potential is regular, i.e. that $|K|<\infty$ globally in $\mathcal{M}_{N_V}$ (or *at least* at the critical points of V_{BH}), one gets that

$$\partial_iV_{BH}=0\Leftrightarrow 2\overline{W}D_iW+ie^{K}W_{ijk}g^{j\bar{l}}g^{k\overline{m}}\left(\overline{D}_{\bar{l}}\overline{W}\right)\overline{D}_{\overline{m}}\overline{W}=0. \tag{3.1.8}$$

3.1.1 Classification of Critical Points of V_{BH}

Starting from the general structure of the criticality conditions (3.1.8) and assuming also the *non-degeneracy* (*i.e.* $V_{BH}|_{\partial V_{BH}=0}>0$) *condition*, the critical points of V_{BH} can be classified in three general classes, analyzed in the next three Subsubsubsects..

Supersymmetric ($\frac{1}{2}$-BPS)

The *supersymmetric ($\frac{1}{2}$-BPS)* critical points of V_{BH} are determined by the constraints (sufficient but not necessary conditions for Eqs. (3.1.8))

$$Z \neq 0, D_i Z = 0, \forall i = 1, ..., n_V. \tag{3.1.1.1.1}$$

The horizon ADM squared mass at $\frac{1}{2}$-BPS critical points of V_{BH} saturates the BPS bound:

$$M^2_{ADM,H,\frac{1}{2}-BPS} = V_{BH,\frac{1}{2}-BPS} = \left[|Z|^2 + g^{i\bar{j}} (D_i Z) \overline{D}_{\bar{j}} \overline{Z}\right]_{\frac{1}{2}-BPS} = |Z|^2_{\frac{1}{2}-BPS} > 0. \tag{3.1.1.1.2}$$

Considering the $\mathcal{N} = 2$, $d = 4$ supergravity Lagrangian in a static, spherically symmetric, asymptotically flat extremal BH background, and denoting by ψ and λ^i respectively the gravitino and gaugino fields, it is easy to see that such a Lagrangian contains terms of the form (see the second and third Refs. of [92])

$$\begin{aligned} &Z\psi\psi; \\ &C_{ijk} g^{k\bar{k}} \left(\overline{D}_{\bar{k}} \overline{Z}\right) \lambda^i \lambda^j; \\ &(D_i Z) \lambda^i \psi. \end{aligned} \tag{3.1.1.1.3}$$

Thus, the conditions (3.1.1.1.1) imply the gaugino mass term and the $\lambda\psi$ term to vanish at the $\frac{1}{2}$-BPS critical points of V_{BH} in $\mathcal{M}_{n_V}$. It is interesting to remark that the gravitino "apparent mass" term $Z\psi\psi$ is in general non-vanishing, also when evaluated at the considered $\frac{1}{2}$-BPS attractors; this is ultimately a consequence of the fact that the extremal BH horizon geometry at the $\frac{1}{2}$-BPS (as well as at the non-BPS) attractors is Bertotti-Robinson $AdS_2 \times S^2$ [97, 98, 99].

Non-supersymmetric (non-BPS) with $Z \neq 0$

The *non-supersymmetric (*non-*BPS)* critical points of V_{BH} with non-vanishing central charge are determined by the constraints

$$Z \neq 0, D_i Z \neq 0, \textit{at least} \text{ for some } i \in \{1, ..., n_V\}, \tag{3.1.1.2.1}$$

which, substituted in Eqs. (3.1.8), yield:

$$\begin{array}{c} D_i Z = -\frac{i}{2\overline{Z}} C_{ijk} g^{j\bar{l}} g^{k\bar{m}} \left(\overline{D}_{\bar{l}} \overline{Z}\right) \overline{D}_{\bar{m}} \overline{Z}, \quad \forall i = 1, ..., n_V; \\ \Updownarrow \\ \overline{D}_{\bar{i}} \overline{Z} = \frac{i}{2Z} \overline{C}_{\bar{i}\bar{j}\bar{k}} g^{l\bar{j}} g^{m\bar{k}} (D_l Z) D_m Z, \quad \forall \bar{i} = \bar{1}, ..., \overline{n_V}, \end{array} \tag{3.1.1.2.2}$$

in turn implying that

$$\begin{aligned} &g^{i\bar{i}} (D_i Z) \overline{D}_{\bar{i}} \overline{Z} = -\frac{i}{2\overline{Z}} C_{ijk} g^{i\bar{i}} g^{j\bar{l}} g^{k\bar{m}} \left(\overline{D}_{\bar{i}} \overline{Z}\right) \left(\overline{D}_{\bar{l}} \overline{Z}\right) \overline{D}_{\bar{m}} \overline{Z} = \\ &= \frac{i}{2Z} \overline{C}_{\bar{i}\bar{j}\bar{k}} g^{i\bar{i}} g^{l\bar{j}} g^{m\bar{k}} (D_i Z) (D_l Z) D_m Z. \end{aligned} \tag{3.1.1.2.3}$$

Such critical points are *non-supersymmetric* ones (*i.e.* they do *not* preserve any of the 8 supersymmetry degrees of freedom of the asymptotical Minkowski background), and they correspond to an extremal, non-BPS BH background. They are commonly named *non-BPS $Z \neq 0$ critical points of V_{BH}*.

AEs (3.1.8) and conditions (3.1.1.2.1) imply

$$(C_{ijk})_{non-BPS, Z\neq 0} \neq 0, \quad \text{for some } (i,j,k) \in \{1, ..., n_V\}^3. \tag{3.1.1.2.4}$$

By using Eq. (3.1.1.2.2) and the so-called SK geometry constraints (see the third of Eqs. (2.18)), the horizon ADM squared mass corresponding to non-BPS $Z \neq 0$ critical points of V_{BH} can be elaborated as follows:

$$M^2_{ADM,H,non-BPS,Z\neq 0} = V_{BH,non-BPS,Z\neq 0} = \left[|Z|^2 + g^{i\overline{j}}\left(D_i Z\right)\overline{D}_{\overline{j}}\overline{Z}\right]_{non-BPS,Z\neq 0} =$$

$$= \left\{ |Z|^2 \left[\begin{array}{l} 1 + \frac{1}{4|Z|^4} R_{k\overline{r}n\overline{s}} g^{k\overline{m}} g^{t\overline{r}} g^{n\overline{l}} g^{u\overline{s}} \left(D_t Z\right)\left(D_u Z\right)\left(\overline{D}_{\overline{l}}\overline{Z}\right)\overline{D}_{\overline{m}}\overline{Z} + \\ + \frac{1}{2|Z|^4}\left[g^{i\overline{j}}\left(D_i Z\right)\overline{D}_{\overline{j}}\overline{Z}\right]^2 \end{array} \right] \right\}_{non-BPS,Z\neq 0}. \tag{3.1.1.2.5}$$

As far as $g_{i\overline{j}}$ is strictly positive-definite globally (or *at least* at the non-BPS $Z \neq 0$ critical points of V_{BH}), $M^2_{ADM,H,non-BPS,Z\neq 0}$ does *not* saturate the BPS bound ([9], [14], [16]):

$$\begin{array}{l} M^2_{ADM,H,non-BPS,Z\neq 0} = V_{BH,non-BPS,Z\neq 0} = \\ = \left[|Z|^2 + g^{i\overline{j}}\left(D_i Z\right)\overline{D}_{\overline{j}}\overline{Z}\right]_{non-BPS,Z\neq 0} > |Z|^2_{non-BPS,Z\neq 0}. \end{array} \tag{3.1.1.2.6}$$

Starting from Eq. (3.1.1.2.5), one can introduce and further elaborate the so-called *non-BPS* $Z \neq 0$ *supersymmetry breaking order parameter* as follows:

$$\begin{array}{l} (0 <)\, \mathcal{O}_{non-BPS,Z\neq 0} \equiv \left[\frac{g^{i\overline{j}}(D_i Z)\overline{D}_{\overline{j}}\overline{Z}}{|Z|^2}\right]_{non-BPS,Z\neq 0} = \\ = -\left[\frac{i}{2\overline{Z}|Z|^2} C_{ijk} g^{i\overline{i}} g^{j\overline{l}} g^{k\overline{m}} \left(\overline{D}_{\overline{i}}\overline{Z}\right)\left(\overline{D}_{\overline{l}}\overline{Z}\right)\overline{D}_{\overline{m}}\overline{Z}\right]_{non-BPS,Z\neq 0} = \\ = \left[\frac{i}{2Z|Z|^2}\overline{C}_{\overline{ijk}} g^{i\overline{i}} g^{l\overline{j}} g^{m\overline{k}} \left(D_i Z\right)\left(D_l Z\right) D_m Z\right]_{non-BPS,Z\neq 0}, \end{array} \tag{3.1.1.2.7}$$

where Eqs. (3.1.1.2.3) were used. Since it holds that

$$\left[\frac{g^{i\overline{j}}(D_i Z)\overline{D}_{\overline{j}}\overline{Z}}{|Z|^2}\right]_{non-BPS,Z\neq 0} =$$

$$= \left\{\frac{1}{4|Z|^4} R_{k\overline{r}n\overline{s}} g^{k\overline{m}} g^{t\overline{r}} g^{n\overline{l}} g^{u\overline{s}} \left(D_t Z\right)\left(D_u Z\right)\left(\overline{D}_{\overline{l}}\overline{Z}\right)\overline{D}_{\overline{m}}\overline{Z} + \frac{1}{2}\left[\frac{g^{i\overline{j}}(D_i Z)\overline{D}_{\overline{j}}\overline{Z}}{|Z|^2}\right]^2\right\}_{non-BPS,Z\neq 0}, \tag{3.1.1.2.8}$$

$\mathcal{O}_{non-BPS,Z\neq 0}$ defined by Eq. (3.1.1.2.7) can equivalently be rewritten as follows:

$$\mathcal{O}_{non-BPS,Z\neq 0} = \left[\frac{1}{4|Z|^4} g^{i\overline{j}} C_{ikn}\overline{C}_{\overline{jrs}} g^{n\overline{l}} g^{k\overline{m}} g^{t\overline{r}} g^{u\overline{s}} \left(D_t Z\right)\left(D_u Z\right)\left(\overline{D}_{\overline{l}}\overline{Z}\right)\overline{D}_{\overline{m}}\overline{Z}\right]_{non-BPS,Z\neq 0}. \tag{3.1.1.2.9}$$

Eqs. (3.1.1.2.9) imply that

$$M^2_{ADM,H,non-BPS,Z\neq 0} = V_{BH,non-BPS,Z\neq 0} = |Z|^2_{non-BPS,Z\neq 0}\left[1 + \mathcal{O}_{non-BPS,Z\neq 0}\right] =$$

$$= \left\{|Z|^2\left[3 - 2\frac{\mathcal{R}(Z)}{g^{i\overline{j}} C_{ikn}\overline{C}_{\overline{jrs}} g^{n\overline{l}} g^{k\overline{m}} g^{t\overline{r}} g^{u\overline{s}} (D_t Z)(D_u Z)\left(\overline{D}_{\overline{l}}\overline{Z}\right)\overline{D}_{\overline{m}}\overline{Z}}\right]\right\}_{non-BPS,Z\neq 0}, \tag{3.1.1.2.10}$$

where the *sectional curvature* (see *e.g.* [100] and [101])

$$\mathcal{R}(Z) \equiv R_{i\bar{j}k\bar{l}} g^{i\bar{i}} g^{j\bar{j}} g^{k\bar{k}} g^{l\bar{l}} (D_j Z)(D_l Z)\left(\overline{D}_{\bar{i}}\overline{Z}\right)\overline{D}_{\bar{k}}\overline{Z} \tag{3.1.1.2.11}$$

was introduced.

Now, by using the relations of SK geometry it is possible to show that

$$\overline{D}_{\overline{m}} D_i C_{jkl} = \left[\overline{D}_{\overline{m}}, D_i\right] C_{jkl} = \overline{D}_{\overline{m}} D_{(i} C_{j)kl} = \overline{D}_{\overline{m}} D_{(i} C_{jkl)} = 3 C_{p(kl} C_{ij)n} g^{n\overline{n}} g^{p\overline{p}} \overline{C}_{\overline{npm}} - 4 g_{(l|\overline{m}} C_{|ijk)}$$

$$\Updownarrow$$

$$C_{p(kl} C_{ij)n} g^{n\overline{n}} g^{p\overline{p}} \overline{C}_{\overline{npm}} = \tfrac{4}{3} g_{(l|\overline{m}} C_{|ijk)} + \overline{E}_{\overline{m}(ijkl)}, \tag{3.1.1.2.12}$$

where we introduced the rank-5 tensor

$$\begin{aligned} \overline{E}_{\overline{m}ijkl} = \overline{E}_{\overline{m}(ijkl)} &\equiv \tfrac{1}{3}\overline{D}_{\overline{m}} D_i C_{jkl} = \tfrac{1}{3}\overline{D}_{\overline{m}} D_{(i} C_{jkl)} = C_{p(kl} C_{ij)n} g^{n\overline{n}} g^{p\overline{p}} \overline{C}_{\overline{npm}} - \tfrac{4}{3} g_{(l|\overline{m}} C_{|ijk)} = \\ &= g^{n\overline{n}} R_{(i|\overline{m}|j|\overline{n}} C_{n|kl)} + \tfrac{2}{3} g_{(i|\overline{m}} C_{|jkl)}, \end{aligned} \tag{3.1.1.2.13}$$

where the SK geometry constraints were used, as well. Now, by recalling the criticality conditions (3.1.8) of V_{BH}, and by using Eq. (3.1.1.2.2), one gets that at non-BPS, $Z \neq 0$ critical points of V_{BH} it holds that

$$\begin{aligned} & 2\overline{Z} D_i Z = \\ & = \tfrac{i}{4Z^2} E_{i\left(\overline{sntu}\right)} g^{p\overline{n}} g^{q\overline{s}} g^{r\overline{t}} g^{v\overline{u}} (D_p Z)(D_q Z)(D_r Z) D_v Z + \\ & + \tfrac{i}{3Z^2} (D_i Z) \overline{C}_{\overline{ntu}} g^{p\overline{n}} g^{r\overline{t}} g^{v\overline{u}} (D_p Z)(D_r Z) D_v Z. \end{aligned} \tag{3.1.1.2.14}$$

By using Eqs. (3.1.1.2.7), (3.1.1.2.14) and (3.1.1.2.7), with a little effort it is thus possible to compute that

$$\begin{aligned} & M^2_{ADM,H,non-BPS,Z\neq 0} = V_{BH,non-BPS,Z\neq 0} = |Z|^2_{non-BPS,Z\neq 0}\left[1 + \mathcal{O}_{non-BPS,Z\neq 0}\right] = \\ & = |Z|^2_{non-BPS,Z\neq 0}\left\{4 - \tfrac{3}{4}\left[\tfrac{1}{|Z|^2}\frac{E_{i(\overline{klmn})} g^{i\overline{j}} g^{k\overline{k}} g^{l\overline{l}} g^{m\overline{m}} g^{n\overline{n}} \left(\overline{D}_{\overline{j}}\overline{Z}\right)(D_k Z)(D_l Z)(D_m Z) D_n Z}{N_3(Z)}\right]_{non-BPS,Z\neq 0}\right\}, \end{aligned} \tag{3.1.1.2.15}$$

where we defined the complex cubic form

$$N_3(Z) \equiv \overline{C}_{\overline{ijk}} g^{i\overline{i}} g^{j\overline{j}} g^{k\overline{k}} (D_i Z)(D_j Z) D_k Z. \tag{3.1.1.2.16}$$

Now, by comparing Eq. (3.1.1.2.15) with Eq. (3.1.1) and by recalling the definition (3.1.1.2.11), one

obtains the following relations to hold at the non-BPS, $Z \neq 0$ critical points of V_{BH}:

$$\frac{3}{4}\left[\frac{1}{|Z|^2}\frac{E_{i(\overline{klmn})}g^{i\overline{j}}g^{k\overline{k}}g^{l\overline{l}}g^{m\overline{m}}g^{n\overline{n}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)(D_kZ)(D_lZ)(D_mZ)D_nZ}{N_3(Z)}\right]_{non-BPS,Z\neq 0}-1=$$

$$=\left[\frac{\mathcal{R}(Z)}{2|Z|^2g^{t\overline{u}}(D_tZ)\overline{D}_{\overline{u}}\overline{Z}}\right]_{non-BPS,Z\neq 0}=$$

$$=2\left[\frac{\mathcal{R}(Z)}{g^{i\overline{j}}C_{ikn}\overline{C}_{\overline{jrs}}g^{n\overline{l}}g^{k\overline{m}}g^{t\overline{r}}g^{u\overline{s}}(D_tZ)(D_uZ)\left(\overline{D}_{\overline{l}}\overline{Z}\right)\overline{D}_{\overline{m}}\overline{Z}}\right]_{non-BPS,Z\neq 0}. \tag{3.1.1.2.17}$$

Let us now consider the case of homogeneous symmetric SK manifolds, in which the Kähler-invariant Riemann-Christoffel tensor $R_{i\overline{j}k\overline{l}}$ is covariantly constant[8]. From this it follows that [88]:

$$D_mR_{i\overline{j}k\overline{l}}=0\Leftrightarrow D_iC_{jkl}=D_{(i}C_{j)kl}=0\Rightarrow \overline{D}_{\overline{m}}D_iC_{jkl}=0\Leftrightarrow D_m\overline{D}_{\overline{i}}\overline{C}_{\overline{jkl}}=0. \tag{3.1.1.2.18}$$

This implies the *global* vanishing of the tensor $\overline{E}_{\overline{i}jklm}$, yielding [88]

$$C_{p(kl}C_{ij)n}g^{n\overline{n}}g^{p\overline{p}}\overline{C}_{\overline{npm}}=\frac{4}{3}g_{(l|\overline{m}}C_{|ijk)}\Leftrightarrow g^{n\overline{n}}R_{(i|\overline{m}|j|\overline{n}}C_{n|kl)}=-\frac{2}{3}g_{(i|\overline{m}}C_{|jkl)}. \tag{3.1.1.2.19}$$

By recalling Eqs. (3.1.1.2.14) and (3.1.1.2.16), one obtains the following noteworthy relation, holding in homogeneous symmetric SK manifolds:

$$\left(Z\left|Z\right|^2\right)_{non-BPS,Z\neq 0}=\frac{i}{6}\left[N_3\left(Z\right)\right]_{non-BPS,Z\neq 0}, \tag{3.1.1.2.20}$$

implying that $\left[\frac{N_3(Z)}{Z}\right]_{non-BPS,Z\neq 0}$ has vanishing real part and strictly negative imaginary part, given by $-6\left|Z\right|^2_{non-BPS,Z\neq 0}$. By recalling Eq. (3.1.1.2.9), Eq. (3.1.1.2.20) implies the value of the *supersymmetry breaking order parameter* at non-BPS, $Z \neq 0$ critical points of V_{BH} in homogeneous symmetric SK manifolds to be

$$\mathcal{O}_{non-BPS,Z\neq 0}=3\Longrightarrow \Delta_{non-BPS,Z\neq 0}=0. \tag{3.1.1.2.21}$$

By recalling Eq. (3.1.1.2.15), one thus finally gets that

$$M^2_{ADM,H,non-BPS,Z\neq 0}=V_{BH,non-BPS,Z\neq 0}=4\left|Z\right|^2_{non-BPS,Z\neq 0}=\frac{2}{3}i\left[\frac{N_3\left(Z\right)}{Z}\right]_{non-BPS,Z\neq 0}, \tag{3.1.1.2.22}$$

where in the last step we used the relation (3.1.1.2.20). The result $V_{BH,non-BPS,Z\neq 0}=4\left|Z\right|^2_{non-BPS,Z\neq 0}$ has been firstly obtained, by exploiting group-theoretical methods, in [21].

Finally, by recalling Eq. (3.1.1.2.17) and using Eqs. (3.1.1.2.20) and (3.1.1.2.22), one obtains the following relation, holding for homogeneous symmetric SK manifolds:

$$\mathcal{R}\left(Z\right)\Big|_{non-BPS,Z\neq 0}=-6\left|Z\right|^4_{non-BPS,Z\neq 0}. \tag{3.1.1.2.23}$$

[8]Indeed, due to the reality of $R_{i\overline{j}k\overline{l}}$ in any Kähler manifold, it holds that

$$D_mR_{i\overline{j}k\overline{l}}=0\Leftrightarrow \overline{D}_{\overline{m}}R_{i\overline{j}k\overline{l}}=0.$$

It is worth pointing out that, while Eq. (3.1.1.2.18) (holding globally) are peculiar to homogeneous symmetric SK manifolds, Eqs. (3.1.1.2.20)-(3.1.1.2.23) hold in general also for homogeneous non-symmetric SK manifolds, in which the Riemann-Christoffel tensor $R_{i\bar{j}k\bar{l}}$ (and thus, through the SK constraints, C_{ijk}) is *not* covariantly constant. Indeed, as obtained in [28] for all the considered non-BPS, $Z \neq 0$ critical points of V_{BH} in homogeneous non-symmetric SK manifolds it holds that

$$\left[E_{i(\overline{klmn})} g^{i\bar{j}} g^{k\bar{k}} g^{l\bar{l}} g^{m\overline{m}} g^{n\bar{n}} \left(\overline{D}_{\bar{j}}\overline{Z}\right) (D_k Z)(D_l Z)(D_m Z) D_n Z\right]_{non-BPS, Z\neq 0} = 0, \tag{3.1.1.2.24}$$

which actually seems to be the most general (necessary and sufficient) condition in order for Eqs. (3.1.1.2.20)-(3.1.1.2.23) to hold. Finally, it should be stressed that in [10] the result (3.1.1.2.21) and thus $V_{BH,non-BPS,Z\neq 0} = 4\left|Z\right|^2_{non-BPS,Z\neq 0}$ was obtained for a generic SK geometry with a cubic holomorphic prepotential (corresponding to the large volume limit of Type IIA on Calabi-Yau threefolds), at least for the non-BPS, $Z \neq 0$ critical points of V_{BH} satisfying the Ansatz

$$z^i_{non-BPS,Z\neq 0} = p^i t(p,q), \ \forall i = 1, ..., n_V, \tag{3.1.1.2.25}$$

where the $z^i_{non-BPS,Z\neq 0}$s are the critical moduli, and $t(p,q)$ is a purely charge-dependent quantity.

Furthermore, it is worth noticing that the general criticality conditions (3.1.5) of V_{BH} can be recognized to be the general Ward identities relating the gravitino mass Z, the gaugino masses D_iD_jZ and the supersymmetry-breaking order parameters D_iZ in a generic spontaneously broken supergravity theory [102]. Indeed, away from $\frac{1}{2}$-BPS critical points (*i.e.* for $D_iZ \neq 0$ for some i), the AEs (3.1.5) can be re-expressed as follows (see also [32]):

$$\left(\mathbf{M}_{ij} h^j\right)_{\partial V_{BH}=0} = 0, \tag{3.1.1.2.26}$$

with

$$\mathbf{M}_{ij} \equiv D_iD_jZ + 2\frac{\overline{Z}}{\left[g^{k\bar{k}}(D_kZ)\left(\overline{D}_{\bar{k}}\overline{Z}\right)\right]}(D_iZ)(D_jZ), \ (\text{Kähler weights } (1,-1)), \tag{3.1.1.2.27}$$

and

$$h^j \equiv g^{j\bar{j}}\overline{D}_{\bar{j}}\overline{Z}, \ (\text{Kähler weights } (-1,1)). \tag{3.1.1.2.28}$$

For a non-vanishing contravariant vector h^j (*i.e.* away from $\frac{1}{2}$-BPS critical points, as pointed out above), Eq. (3.1.1.2.26) admits a solution iff the $n_V \times n_V$ complex symmetric matrix $\mathbf{M}_{ij}$ has vanishing determinant (implying that it has at most $n_V - 1$ non-vanishing eigenvalues) at the considered (non-BPS) critical points of V_{BH} (however, notice that $\mathbf{M}_{ij}$ is symmetric but not necessarily Hermitian, thus in general its eigenvalues are not necessarily real). Such a reasoning holds for all non-BPS critical points of V_{BH}, *i.e.* for the classes II and III of the presented classification.

In general, non-BPS $Z \neq 0$ critical points of V_{BH} in $\mathcal{M}_{n_V}$ are not necessarily stable, because the $2n_V \times 2n_V$ (covariant) Hessian matrix (in $(z,\overline{z})$-coordinates) of V_{BH} evaluated at such points is not necessarily strictly positive-definite. An explicit condition of stability of non-BPS $Z \neq 0$ critical points of V_{BH} can be worked out in the $n_V = 1$ case (see [17], [18], [26]).

In general, Eqs. (3.1.1.1.3) and conditions (3.1.1.2.1) imply the gaugino mass term, the $\lambda\psi$ term and the gravitino "apparent mass" term $Z\psi\psi$ to be non-vanishing, when evaluated at the considered non-BPS $Z \neq 0$ critical points of V_{BH}.

Non-supersymmetric (non-BPS) with $Z = 0$

The *non-supersymmetric (non-BPS)* critical points of V_{BH} with vanishing central charge are determined by the constraints

$$Z = 0, D_i Z \overset{Z=0}{=} \partial_i Z \neq 0, \textit{at least} \text{ for some } i \in \{1, ..., n_V\}, \tag{3.1.1.3.1}$$

which, substituted in Eqs. (3.1.8), yield:

$$C_{ijk} g^{j\bar{l}} g^{k\overline{m}} \left(\overline{D_{\bar{l}} Z}\right) \overline{D_{\overline{m}} Z} \overset{Z=0}{=} C_{ijk} g^{j\bar{l}} g^{k\overline{m}} \left(\overline{\partial}_{\bar{l}} \overline{Z}\right) \overline{\partial}_{\overline{m}} \overline{Z} = 0, \quad \forall i = 1, ..., n_V. \tag{3.1.1.3.2}$$

Such critical points are *non-supersymmetric* ones, but, differently from the class II considered above, they correspond to an extremal, non-BPS BH background in which the horizon $\mathcal{N} = 2$, $d = 4$ supersymmetry algebra is not centrally extended. They are commonly named *non-BPS $Z = 0$ critical points of V_{BH}*.

The horizon ADM squared mass corresponding to non-BPS $Z = 0$ critical points of V_{BH} does *not* saturate the BPS bound ([9], [14], [16]):

$$\begin{aligned} & M^2_{ADM,H,non-BPS,Z=0} = V_{BH,non-BPS,Z=0} = \\ & = \left\{ g^{i\bar{j}} \left(\partial_i Z\right) \overline{\partial}_{\bar{j}} \overline{Z} \right\}_{non-BPS,Z=0} > \left(|Z|^2\right)_{non-BPS,Z=0} = 0, \end{aligned} \tag{3.1.1.3.3}$$

as far as $g_{i\bar{j}}$ is strictly positive-definite globally (or *at least* at the considered critical points of V_{BH}). Eqs. (3.1.1.3.2) suggest the following sub-classification of non-BPS $Z = 0$ critical points of V_{BH}:

III.1) Critical points determined by the conditions

$$\begin{cases} Z = 0, \\ D_i Z \overset{Z=0}{=} \partial_i Z \neq 0, \textit{at least} \text{ for some } i \in \{1, ..., n_V\}, \\ C_{ijk} = 0, \forall i, j, k, \end{cases} \tag{3.1.1.3.4}$$

directly solving Eqs. (3.1.1.3.2) and thus AEs (3.1.8). This is the only possible case for $n_V = 1$.

In particular, non-BPS $Z = 0$ critical points of V_{BH} do not exist at all in the $n_V = 1$ case of the so-called "*d-SK geometries*", whose stringy origin is *e.g.* Type IIA on CY_3 in the large volume limit of CY_3 (see e.g. [10]). Indeed, in such a case in special projective coordinates (with Kähler gauge fixed such that $X^0 \equiv 1$) the holomorphic prepotential $\mathcal{F}$ and W_{ijk} respectively read

$$\begin{aligned} & \mathcal{F} = d_{ijk} z^i z^j z^k; \\ & C_{ijk} = e^K d_{ijk}, \end{aligned} \tag{3.1.1.3.5}$$

and thus, for $|K| < \infty$ at least at the considered critical points of V_{BH}, the third of conditions (3.1.1.3.4) cannot be satisfied.

III.2) Critical points determined by the conditions

$$\begin{cases} Z = 0, \\ D_i Z \overset{Z=0}{=} \partial_i Z \neq 0, \quad \textit{at least} \text{ for some } i \in \{1, ..., n_V\}, \\ C_{ijk} \neq 0, \quad \textit{at least} \text{ for some } (i, j, k) \in \{1, ..., n_V\}^3, \end{cases} \tag{3.1.1.3.6}$$

for which Eqs. (3.1.1.3.2), and thus AEs (3.1.8), are not trivially solved.

In general, non-BPS $Z = 0$ critical points of V_{BH} in $\mathcal{M}_{n_V}$ are not necessarily stable, because the $2n_V \times 2n_V$ (covariant) Hessian matrix (in $(z, \overline{z})$-coordinates) of V_{BH} evaluated at such points is not necessarily strictly positive-definite. An explicit condition of stability of non-BPS $Z = 0$ critical points of V_{BH} can be worked out in the $n_V = 1$ case [26].

In general, Eqs. (3.1.1.1.3) and conditions (3.1.1.3.1) imply the the $\lambda\psi$ term to be non-vanishing and the gravitino "apparent mass" term $Z\psi\psi$ to vanish, when evaluated at the considered non-BPS $Z = 0$ critical points of V_{BH}, characterized by vanishing (class III.1) or non-vanishing (class III.2) gaugino mass terms.

Non-BPS $Z = 0$ attractors in the so-called st^2 and stu models [103, 104] have been recently studied in [43], and their relation with the $\frac{1}{2}$-BPS attractors has been analyzed in light of the uplift to $\mathcal{N} = 8$, $d = 4$ supergravity.

3.2 Stability of Critical Points of V_{BH}

3.2.1 n_V-Moduli

In order to decide whether a critical point of V_{BH} is an attractor in strict sense, one has to consider the following condition:

$$H_{\mathbb{R}}^{V_{BH}} \equiv H_{ab}^{V_{BH}} \equiv D_a D_b V_{BH} > 0 \quad \text{at} \quad D_c V_{BH} = \frac{\partial V_{BH}}{\partial \phi^c} = 0 \quad \forall c = 1, ..., 2n_V, \tag{3.2.1.1}$$

i.e. the condition of (strict) positive-definiteness of the real $2n_V \times 2n_V$ Hessian matrix $H_{\mathbb{R}}^{V_{BH}} \equiv H_{ab}^{V_{BH}}$ of V_{BH} (which is nothing but the squared mass matrix of the moduli) at the critical points of V_{BH}, expressed in the real parameterization through the ϕ-coordinates. Since V_{BH} is positive-definite, a stable critical point (namely, an attractor in strict sense) is necessarily a(n at least local) minimum, and therefore it fulfills the condition (3.2.1.1).

In general, $H_{\mathbb{R}}^{V_{BH}}$ may be block-decomposed in $n_V \times n_V$ real matrices:

$$H_{\mathbb{R}}^{V_{BH}} = \begin{pmatrix} \mathcal{A} & \mathcal{C} \\ \mathcal{C}^T & \mathcal{B} \end{pmatrix}, \tag{3.2.1.2}$$

with $\mathcal{A}$ and $\mathcal{B}$ being $n_V \times n_V$ real symmetric matrices:

$$\mathcal{A}^T = \mathcal{A}, \ \mathcal{B}^T = \mathcal{B} \Leftrightarrow \left(H_{\mathbb{R}}^{V_{BH}}\right)^T = H_{\mathbb{R}}^{V_{BH}}. \tag{3.2.1.3}$$

In the local complex $(z, \overline{z})$-parameterization, the $2n_V \times 2n_V$ Hessian matrix of V_{BH} reads

$$H_{\mathbb{C}}^{V_{BH}} \equiv H_{\hat{i}\hat{j}}^{V_{BH}} \equiv \begin{pmatrix} D_i D_j V_{BH} & D_i \overline{D}_{\overline{j}} V_{BH} \\ D_j \overline{D}_{\overline{i}} V_{BH} & \overline{D}_{\overline{i}} \overline{D}_{\overline{j}} V_{BH} \end{pmatrix} = \begin{pmatrix} \mathcal{M}_{ij} & \mathcal{N}_{i\overline{j}} \\ \overline{\mathcal{N}_{i\overline{j}}} & \overline{\mathcal{M}_{ij}} \end{pmatrix}, \tag{3.2.1.4}$$

where the hatted indices $\hat{\imath}$ and $\hat{\jmath}$ may be holomorphic or antiholomorphic. $H_{\mathbb{C}}^{V_{BH}}$ is the matrix actually computable in the SKG formalism presented in Sect. 2 (see below, Eqs. (3.2.1.6) and (3.2.1.7)).

In general, $\frac{1}{2}$-BPS critical points are (at least local) minima of V_{BH}, and therefore they are stable; thus, they are *attractors* in strict sense. Indeed, the $2n_V \times 2n_V$ (covariant) Hessian matrix $H_{\mathbb{C}}^{V_{BH}}$

evaluated at such points is strictly positive-definite [5] :

$$\left(D_iD_jV_{BH}\right)_{\frac{1}{2}-BPS}=\left(\partial_i\partial_jV_{BH}\right)_{\frac{1}{2}-BPS}=0,$$
$$\left(D_i\overline{D}_{\overline{j}}V_{BH}\right)_{\frac{1}{2}-BPS}=\left(\partial_i\overline{\partial}_{\overline{j}}V_{BH}\right)_{\frac{1}{2}-BPS}=2\left(g_{i\overline{j}}V_{BH}\right)_{\frac{1}{2}-BPS}=2\left.g_{i\overline{j}}\right|_{\frac{1}{2}-BPS}|Z|^2_{\frac{1}{2}-BPS}>0, \tag{3.2.1.5}$$

where here and below the notation "> 0" ("< 0") is understood as strict positive-(negative-)definiteness. The Hermiticity and (strict) positive-definiteness of the (covariant) Hessian matrix $H^{V_{BH}}_{\mathbb{C}}$ at the $\frac{1}{2}$-BPS critical points are due to the Hermiticity and - assumed - (strict) positive-definiteness (actually holding globally) of the metric $g_{i\overline{j}}$ of the SK scalar manifold being considered.

On the other hand, non-BPS critical points of V_{BH} does not automatically fulfill the condition (3.2.1.1), and a more detailed analysis [21, 18] is needed.

Using the properties of SKG, one obtains:

$$\mathcal{M}_{ij}\equiv D_iD_jV_{BH}=D_jD_iV_{BH}=4i\overline{Z}C_{ijk}g^{k\overline{k}}\left(\overline{D}_{\overline{k}}\overline{Z}\right)+i\left(D_jC_{ikl}\right)g^{k\overline{k}}g^{l\overline{l}}\left(\overline{D}_{\overline{k}}\overline{Z}\right)\left(\overline{D}_{\overline{l}}\overline{Z}\right); \tag{3.2.1.6}$$

$$\mathcal{N}_{i\overline{j}}\equiv D_i\overline{D}_{\overline{j}}V_{BH}=\overline{D}_{\overline{j}}D_iV_{BH}=2\left[g_{i\overline{j}}|Z|^2+\left(D_iZ\right)\left(\overline{D}_{\overline{j}}\overline{Z}\right)+g^{l\overline{n}}C_{ikl}\overline{C}_{\overline{j}\overline{m}\overline{n}}g^{k\overline{k}}g^{m\overline{m}}\left(\overline{D}_{\overline{k}}\overline{Z}\right)\left(D_mZ\right)\right], \tag{3.2.1.7}$$

with D_jC_{ikl} given by Eq. (2.19). Clearly, evaluating Eqs. (3.2.1.6) and (3.2.1.7) constrained by the $\frac{1}{2}$-BPS conditions $D_iZ=0,\forall i=1,...,n_V$, one reobtains the results (3.2.1.5). Here we limit ourselves to point out that further noteworthy elaborations of $\mathcal{M}_{ij}$ and $\mathcal{N}_{i\overline{j}}$ can be performed in homogeneous symmetric SK manifolds, where $D_jC_{ikl}=0$ globally [21], and that the Kähler-invariant $(2,2)$-tensor $g^{l\overline{n}}C_{ikl}\overline{C}_{\overline{j}\overline{m}\overline{n}}$ can be rewritten in terms of the Riemann-Christoffel tensor $R_{i\overline{j}k\overline{m}}$ by using the so-called "SKG constraints" (see the third of Eqs. (2.18)) [18]. Moreover, the differential Bianchi identities for $R_{i\overline{j}k\overline{m}}$ imply $\mathcal{M}_{ij}$ to be symmetric (see comment below Eqs. (2.18) and (2.19)).

Thus, one gets the following global properties:

$$\mathcal{M}^T=\mathcal{M},\;\;\mathcal{N}^\dagger=\mathcal{N}\Leftrightarrow\left(H^{V_{BH}}_{\mathbb{C}}\right)^T=H^{V_{BH}}_{\mathbb{C}}, \tag{3.2.1.8}$$

implying that

$$\left(H^{V_{BH}}_{\mathbb{C}}\right)^\dagger=H^{V_{BH}}_{\mathbb{C}}\Leftrightarrow\mathcal{M}^\dagger=\mathcal{M},\;\;\mathcal{N}^T=\mathcal{N}\Leftrightarrow\overline{\mathcal{M}}=\mathcal{M},\;\;\overline{\mathcal{N}}=\mathcal{N}. \tag{3.2.1.9}$$

It should be stressed clearly that the symmetry but non-Hermiticity of $H^{V_{BH}}_{\mathbb{C}}$ actually does not matter, because what one is ineterested in are the eigenvalues of the real form $H^{V_{BH}}_{\mathbb{R}}$, which is real and symmetric, and therefore admitting $2n_V$ *real* eigenvalues.

The relation between $H^{V_{BH}}_{\mathbb{R}}$ expressed by Eq. (3.2.1.2) and $H^{V_{BH}}_{\mathbb{C}}$ given by Eq. (3.2.1.4) is expressed by the following relations between the $n_V\times n_V$ sub-blocks of $H^{V_{BH}}_{\mathbb{R}}$ and $H^{V_{BH}}_{\mathbb{C}}$ [17, 29]:

$$\left\{\begin{array}{l}\mathcal{M}=\frac{1}{2}\left(\mathcal{A}-\mathcal{B}\right)+\frac{i}{2}\left(\mathcal{C}+\mathcal{C}^T\right);\\[2mm]\mathcal{N}=\frac{1}{2}\left(\mathcal{A}+\mathcal{B}\right)+\frac{i}{2}\left(\mathcal{C}^T-\mathcal{C}\right),\end{array}\right. \tag{3.2.1.10}$$

or its inverse

$$\begin{cases} \mathcal{A} = Re\mathcal{M} + Re\mathcal{N}; \\ \mathcal{B} = Re\mathcal{N} - Re\mathcal{M}; \\ \mathcal{C} = Im\mathcal{M} - Im\mathcal{N}. \end{cases} \tag{3.2.1.11}$$

The structure of the Hessian matrix gets simplified at the critical points of V_{BH}, because the covariant derivatives may be substituted by the flat ones; the critical Hessian matrices in complex holomorphic/antiholomorphic and real local parameterizations respectively read

$$H_{\mathbb{C}}^{V_{BH}}\Big|_{\partial V_{BH}=0} \equiv \begin{pmatrix} \partial_i\partial_j V_{BH} & \partial_i\overline{\partial}_{\overline{j}} V_{BH} \\ \partial_j\overline{\partial}_{\overline{i}} V_{BH} & \overline{\partial}_{\overline{i}}\overline{\partial}_{\overline{j}} V_{BH} \end{pmatrix}_{\partial V_{BH}=0} = \begin{pmatrix} \mathcal{M} & \mathcal{N} \\ \overline{\mathcal{N}} & \overline{\mathcal{M}} \end{pmatrix}_{\partial V_{BH}=0}; \tag{3.2.1.12}$$

$$H_{\mathbb{R}}^{V_{BH}}\Big|_{\partial V_{BH}=0} = \frac{\partial^2 V_{BH}}{\partial\phi^a\partial\phi^b}\Big|_{\partial V_{BH}=0} = \begin{pmatrix} \mathcal{A} & \mathcal{C} \\ \mathcal{C}^T & \mathcal{B} \end{pmatrix}_{\partial V_{BH}=0}. \tag{3.2.1.13}$$

Thus, the study of the condition (3.2.1.1) finally amounts to the study of the *eigenvalue problem* of the real symmetric $2n_V \times 2n_V$ critical Hessian matrix $H_{\mathbb{R}}^{V_{BH}}\Big|_{\partial V_{BH}=0}$ given by Eq. (3.2.1.13), which is the Cayley-transformed of the complex (symmetric, but not necessarily Hermitian) $2n_V \times 2n_V$ critical Hessian $H_{\mathbb{C}}^{V_{BH}}\Big|_{\partial V_{BH}=0}$ given by Eq. (3.2.1.12).

3.2.2 1-Modulus

Once again, the situation strongly simplifies in $n_V = 1$ SKG.

Indeed, for $n_V = 1$ the moduli-dependent matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{M}$ and $\mathcal{N}$ introduced above are simply scalar functions. In particular, $\mathcal{N}$ is real, since $\mathcal{C}$ trivially satisfies $\mathcal{C} = \mathcal{C}^T$. The stability condition (3.2.1.1) can thus be written as

$$H_{\mathbb{R}}^{V_{BH}} \equiv D_a D_b V_{BH} > 0,\ (a, b = 1, 2) \quad \text{at} \quad D_c V_{BH} = \frac{\partial V_{BH}}{\partial\phi^c} = 0 \quad \forall c = 1, 2, \tag{3.2.2.1}$$

and Eqs. (3.2.1.6) and (3.2.1.7) respectively simplify to

$$\mathcal{M} \equiv D^2 V_{BH} = 4i\overline{Z}Cg^{-1}\overline{DZ} + i\,(DC)\,g^{-2}\left(\overline{DZ}\right)^2; \tag{3.2.2.2}$$

$$\mathcal{N} \equiv D\overline{D}V_{BH} = \overline{D}DV_{BH} = 2\left[g\,|Z|^2 + |DZ|^2 + |C|^2 g^{-3}\,|DZ|^2\right], \tag{3.2.2.3}$$

DC being given by the case $n_V = 1$ of Eq. (2.19):

$$DC = \partial C + [(\partial K) + 3\Gamma]\,C = \partial C + [(\partial K) - 3\partial ln\,(g)]\,C = \left\{\partial + \left[\partial ln\left(\frac{e^K}{\left(\overline{\partial}\partial K\right)^3}\right)\right]\right\}C, \tag{3.2.2.4}$$

where the $n_V = 1$ Christoffel connection

$$\Gamma = -g^{-1}\partial g = -\partial ln\,(g) \tag{3.2.2.5}$$

was used. It is easy to show that the stability condition (3.2.2.1) for critical points of V_{BH} in $n_V = 1$ SKG can be equivalently reformulated as the strict bound

$$\mathcal{N}|_{\partial V_{BH}=0} > |\mathcal{M}|_{\partial V_{BH}=0}\,. \tag{3.2.2.6}$$

Let us now see how such a bound can be further elaborated for the three possible classes of critical points of V_{BH}.

$\frac{1}{2}$-BPS

$$\mathcal{M}_{\frac{1}{2}-BPS} \equiv D^2 V_{BH}\big|_{\frac{1}{2}-BPS} = \left[3\overline{Z}D^2 Z + g^{-1}\left(D^3 Z\right)\overline{DZ}\right]_{\frac{1}{2}-BPS} = 0; \tag{3.2.2.1.1}$$

$$\mathcal{N}_{\frac{1}{2}-BPS} \equiv D\overline{D}V_{BH}\big|_{\frac{1}{2}-BPS} = \left[2g\left|Z\right|^2 + g^{-1}\left|D^2 Z\right|^2\right]_{\frac{1}{2}-BPS} = 2\left(g\left|Z\right|^2\right)_{\frac{1}{2}-BPS}. \tag{3.2.2.1.2}$$

Eqs. (3.2.2.1.1) and (3.2.2.1.2) are nothing but the 1-modulus case of Eq. (3.2.1.5), and they directly satisfy the bound (3.2.2.6). Thus, consistently with what stated above, the $\frac{1}{2}$-BPS class of critical points of V_{BH} actually is a class of attractors in strict sense (*at least* local minima of V_{BH}).

Non-BPS, $Z \neq 0$

$$\begin{aligned}
&\mathcal{M}_{non-BPS,Z\neq 0} \equiv D^2 V_{BH}\big|_{non-BPS,Z\neq 0} = \\
&= -2\left\{g^{-1}\overline{Z}DZ\left[g^{-2}\left|C\right|^2 Dln\left(Z\right) + gDln\left(C\right)\right]\right\}_{non-BPS,Z\neq 0} = \\
&= i\left\{Cg^{-3}\left(\overline{DZ}\right)^2\left[g^{-2}\left|C\right|^2 Dln\left(Z\right) + gDln\left(C\right)\right]\right\}_{non-BPS,Z\neq 0};
\end{aligned} \tag{3.2.2.2.1}$$

$$\begin{aligned}
&\mathcal{N}_{non-BPS,Z\neq 0} \equiv D\overline{D}V_{BH}\big|_{non-BPS,Z\neq 0} = \overline{D}DV_{BH}\big|_{non-BPS,Z\neq 0} = \\
&= 2\left\{\left|DZ\right|^2\left[1 + \tfrac{5}{4}g^{-3}\left|C\right|^2\right]\right\}_{non-BPS,Z\neq 0}.
\end{aligned} \tag{3.2.2.2.2}$$

Eq. (3.2.2.2.1) yields that

$$\begin{aligned}
&\left|\mathcal{M}\right|^2_{non-BPS,Z\neq 0} = \\
&= 4\left\{\left|DZ\right|^4\left[\left|C\right|^4 g^{-6} + \tfrac{1}{4}g^{-4}\left|DC\right|^2 + 2g^{-3}Re\left[C\left(\overline{DC}\right)Dln\left(Z\right)\right]\right]\right\}_{non-BPS,Z\neq 0}.
\end{aligned} \tag{3.2.2.2.3}$$

By substituting Eqs. (3.2.2.2.2) and (3.2.2.2.3) into the strict inequality (3.2.2.6), one finally obtains the stability condition for non-BPS, $Z \neq 0$ critical points of V_{BH} in $n_V = 1$ SKG [17]:

$$\mathcal{N}_{non-BPS,Z\neq 0} > |\mathcal{M}|_{non-BPS,Z\neq 0}\,; \tag{3.2.2.2.4}$$

$$\Updownarrow$$

$$1+\frac{5}{4}\left(|C|^2 g^{-3}\right)_{non-BPS,Z\neq 0} > \sqrt{\left[|C|^4 g^{-6}+\frac{1}{4}g^{-4}|DC|^2+2g^{-3}Re\left[C\left(\overline{DC}\right)\left(\overline{D}ln\overline{Z}\right)\right]\right]_{non-BPS,Z\neq 0}}\,. \tag{3.2.2.2.5}$$

As it is seen from such a condition, in general $(DC)_{non-BPS,Z\neq 0}$ is the fundamental geometrical quantity playing a key role in determining the stability of non-BPS, $Z \neq 0$ critical points of V_{BH} in 1-modulus SK geometry.

Non-BPS, $Z = 0$

$$\mathcal{M}_{non-BPS,Z=0} \equiv D^2 V_{BH}\Big|_{non-BPS,Z=0} = i\left[g^{-2}\left(\partial C\right)\left(\overline{\partial Z}\right)^2\right]_{non-BPS,Z=0}\,; \tag{3.2.2.3.1}$$

$$\mathcal{N}_{non-BPS,Z=0} \equiv D\overline{D}V_{BH}\Big|_{non-BPS,Z=0} = 2\left|\partial Z\right|^2_{non-BPS,Z=0}\,, \tag{3.2.2.3.2}$$

where Eqs. (3.1.1.3.6) and (3.1.1.3.4) have been used. Eq. (3.2.2.3.1) yields that

$$|\mathcal{M}|_{non-BPS,Z=0} = \left[g^{-2}\left|\partial C\right|\left|\partial Z\right|^2\right]_{non-BPS,Z=0}. \tag{3.2.2.3.3}$$

By substituting Eqs. (3.2.2.3.2) and (3.2.2.3.3) into the strict inequality (3.2.2.6), one finally obtains the stability condition for non-BPS, $Z = 0$ critical points of V_{BH} in $n_V = 1$ SKG:

$$\mathcal{N}_{non-BPS,Z=0} > |\mathcal{M}|_{non-BPS,Z=0}\,; \tag{3.2.2.3.4}$$

$$\Updownarrow$$

$$2g^2_{non-BPS,Z=0} > |\partial C|_{non-BPS,Z=0}\,. \tag{3.2.2.3.5}$$

Even though the stability condition (3.2.2.3.5) have been obtained by correctly using Eqs. (3.1.1.3.6) and (3.1.1.3.4), holding at the non-BPS, $Z = 0$ critical points of V_{BH}, in some cases it may happen that, in the limit of approaching the non-BPS, $Z = 0$ critical point of V_{BH}, in DC (given by Eq. (3.2.2.4)) the "connection term" $[(\partial K) + 3\Gamma]\,C$ is not necessarily sub-leading with respect to the "differential term" ∂C. Thus, the condition (3.2.2.3.5) can be rewritten as follows:

$$\begin{aligned} 2\left(\overline{\partial}\partial K\right)^2_{non-BPS,Z=0} &> \left|\left\{\partial + \left[(\partial K) - 3\partial ln\left(g\right)\right]\right\}C\right|_{non-BPS,Z=0} = \\ &= \left|\left\{\partial + \left[\partial ln\left(\frac{e^K}{\left(\overline{\partial}\partial K\right)^3}\right)\right]\right\}C\right|_{non-BPS,Z=0}. \end{aligned} \tag{3.2.2.3.6}$$

Remark

Let us consider the 1-modulus stability conditions (3.2.2.2.4) and (3.2.2.3.5)-3.2.2.3.6). It is immediate to realize that they are both satisfied when the function C is globally covariantly constant:

$$DC = \partial C + \left[(\partial K) + 3\Gamma \right] C = 0, \tag{3.2.2.4.1}$$

i.e. for the so-called homogeneous symmetric ($dim_{\mathbb{C}} = n_V = 1$) SK geometry [87, 88], univoquely associated to the coset manifold $\frac{SU(1,1)}{U(1)}$. Such a SK manifold can be twofold characterized as:

i) the $n = 0$ element of the irreducible rank-1 infinite sequence $\frac{SU(1,1+n)}{U(1) \otimes SU(1+n)}$ (with $n_V = n + rank = n+1$), or equivalently the $n = -2$ element of the reducible rank-3 infinite sequence $\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,2+n)}{SO(2) \otimes SO(2+n)}$ (with $n_V = n + rank = n + 3$). In such a case, $\frac{SU(1,1)}{U(1)}$ is endowed with a quadratic holomorphic prepotential function reading (in a suitable projective special coordinate, with Kähler gauge fixed such that $X^0 = 1$; see [21] and Refs. therein)

$$\mathcal{F}(z) = \frac{i}{2} \left(z^2 - 1 \right). \tag{3.2.2.4.2}$$

By recalling the first of Eqs. (2.18), such a prepotential yields $C = 0$ globally (and thus Eq. (3.2.2.4.1)), and therefore, by using the SKG constraints (i.e. the third of Eqs. (2.18)) it yields also the constant scalar curvature

$$\mathcal{R} \equiv g^{-2} R = -2, \tag{3.2.2.4.3}$$

where $R \equiv R_{1\bar{1}1\bar{1}}$ denotes the unique component of the Riemann tensor. As obtained in [21], quadratic (homogeneous symmetric) SK geometries only admit $\frac{1}{2}$-BPS and non-BPS, $Z = 0$ critical points of V_{BH}. Thus, it can be concluded that the 1-dim. quadratic SK geometry determined by the prepotential (3.2.2.4.2) admits *all* stable critical points of V_{BH}.

ii) the rank-1 $s = t = u \equiv z$ *degeneration* of the so-called *stu* model [103, 104] ($n = 0$ element of the reducible rank-3 infinite sequence $\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,2+n)}{SO(2) \otimes SO(2+n)}$), or equivalently the rank-1 $s = t \equiv z$ *degeneration* of the so-called st^2 model ($n = -1$ element of $\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,2+n)}{SO(2) \otimes SO(2+n)}$), or also as an isolated case in the classification of homogeneous symmetric SK manifolds (see *e.g.* [105]). In such a case, $\frac{SU(1,1)}{U(1)}$ is endowed with a cubic holomorphic prepotential function reading (in a suitable projective special coordinate, with Kähler gauge fixed such that $X^0 = 1$; see *e.g.* [21] and Refs. therein)

$$\mathcal{F}(z) = \varrho z^3, \;\; \varrho \in \mathbb{C}, \tag{3.2.2.4.4}$$

constrained by the condition $Im\,(z) < 0$. It admits an uplift to *pure* $\mathcal{N} = 2$ supergravity in $d = 5$. By recalling the first of Eqs. (2.18), such a prepotential yields $C = 6\varrho e^K$ (and thus Eq. (3.2.2.4.1)), and consequently it also yields the constant scalar curvature

$$\mathcal{R} \equiv g^{-2} R = g^{-2} \left(-2g^2 + g^{-1} \left| C \right|^2 \right) = -\frac{2}{3}, \tag{3.2.2.4.5}$$

where the SKG constraints (i.e. the third of Eqs. (2.18)) and the global value[9] $\left| C \right|^2 g^{-3} = \frac{4}{3}$ have been used. As it can be computed (see *e.g.* [32]), the 1-dim. SK geometry determined by the prepotential

[9]The global value $\left| C \right|^2 g^{-3} = \frac{4}{3}$ for homogeneous symmetric cubic $n_V = 1$ SK geometries is yielded by the $n_V = 1$ case of Eq. (3.1.1.2.19).

(3.2.2.4.4) admits, beside the (stable) $\frac{1}{2}$-BPS ones, stable non-BPS $Z \neq 0$ critical points of V_{BH}. Thus, it is another example in which *all* critical points of V_{BH} actually are attractors in a strict sense.

Clearly, the quadratic and cubic homogeneous symmetric 1-modulus SK geometries (respectively determined by holomorphic prepotentials (3.2.2.4.2) and (3.2.2.4.4)) are not the only ones (with $n_V = 1$) admitting stable non-BPS critical points of V_{BH}. For instance, as studied in [26], the 1-modulus SK geometries of the moduli space of the (mirror) Fermat CY_3 *quintic* $\mathcal{M}'_5$ and *octic* $\mathcal{M}'_8$ admit, in a suitable neighbourhood of the LG point, stable non-BPS ($Z \neq 0$) critical points of V_{BH}.

It is worth remarking that recent works [33, 36, 46] gave a complete treatment of the issue of stability of non-BPS attractors in the framework of homogeneous SKGs, finding that the massless modes of the non-BPS Hessian matrix actually are "flat directions" of V_{BH} at the considered class of critical points. This means that non-BPS attractors in $\mathcal{N} = 2$, $d = 4$ supergravity have a related moduli space, spanned by those moduli which are not stabilized at the BH horizon. However, it should be pointed out that such an emergence of moduli spaces do not violate the Attractor Mechanism and/or the determinacy of BH thermodinamical properties, because the non-BPS BH entropy simply does not depend on the scalar degrees of freedom spanning the moduli space of the considered class ($Z \neq 0$ or $Z = 0$) of non-BPS critical points of V_{BH}. Such considerations hold also for $\mathcal{N} > 2$-extended, $d = 4$ supergravities (where also BPS attractors can have a related moduli space), and in general in all theories with an homogeneous (not necessarily symmetric) scalar manifold [33, 40, 46, 49].

3.3 $\mathcal{N} = 2$, $d = 4$ General Formulation

3.3.1 Special Kähler Geometry Identities

We will now derive some important identities of the SK geometry [106, 9, 14, 17, 18, 23] of the scalar manifold of $\mathcal{N} = 2$, $d = 4$ ungauged supergravity. Such identities extend the results obtained by Ferrara and Kallosh in [3].

Let us start by considering the covariant antiholomorphic derivative of $\overline{Z}$; by recalling the definition (2.2) and using the second of *Ansätze* (2.21), one gets

$$\overline{D}_{\bar{j}}\overline{Z} = q_\Lambda \overline{D}_{\bar{i}}\overline{L}^\Lambda - p^\Lambda \mathcal{N}_{\Lambda\Delta}\overline{D}_{\bar{j}}\overline{L}^\Delta. \tag{3.3.1.1}$$

The contraction of both sides with $g^{i\bar{j}}D_i L^\Sigma$ then yields

$$g^{i\bar{j}}\left(D_i L^\Sigma\right)\overline{D}_{\bar{j}}\overline{Z} = q_\Lambda g^{i\bar{j}}\left(D_i L^\Sigma\right)\overline{D}_{\bar{j}}\overline{L}^\Lambda - p^\Lambda \mathcal{N}_{\Lambda\Delta} g^{i\bar{j}}\left(D_i L^\Sigma\right)\overline{D}_{\bar{j}}\overline{L}^\Delta. \tag{3.3.1.2}$$

By exploiting the symmetry of $\mathcal{N}_{\Lambda\Sigma}$ and its inverse (see Eq. (2.23) and Eq. (3.4.1.9) further below, as well), recalling the first of the Ansätze (2.21), and using the result of SK geometry (see *e.g.* [64])

$$g^{i\bar{j}}\left(D_i L^\Lambda\right)\overline{D}_{\bar{j}}\overline{L}^\Sigma = -\frac{1}{2}\left(Im\mathcal{N}\right)^{-1|\Lambda\Sigma} - \overline{L}^\Lambda L^\Sigma, \tag{3.3.1.3}$$

Eq.(3.3.1.2) can be further elaborated as follows:

$$g^{i\bar{j}}\left(D_iL^{\Sigma}\right)\overline{D}_{\bar{j}}\overline{Z}=q_{\Lambda}\left[-\tfrac{1}{2}\left(Im\mathcal{N}\right)^{-1|\Sigma\Lambda}-\overline{L}^{\Sigma}L^{\Lambda}\right]-p^{\Lambda}\mathcal{N}_{\Lambda\Delta}\left[-\tfrac{1}{2}\left(Im\mathcal{N}\right)^{-1|\Sigma\Delta}-\overline{L}^{\Sigma}L^{\Delta}\right]=$$

$$=-\tfrac{1}{2}\left(Im\mathcal{N}\right)^{-1|\Sigma\Lambda}q_{\Lambda}-\overline{L}^{\Sigma}\left(L^{\Lambda}q_{\Lambda}-M_{\Lambda}p^{\Lambda}\right)+\tfrac{1}{2}\left(Im\mathcal{N}\right)^{-1|\Sigma\Delta}\left(Re\mathcal{N}_{\Delta\Lambda}\right)p^{\Lambda}+\tfrac{i}{2}p^{\Sigma}=$$

$$=\tfrac{i}{2}p^{\Sigma}-\overline{L}^{\Sigma}Z+\tfrac{1}{2}\left(Im\mathcal{N}\right)^{-1|\Sigma\Delta}\left(Re\mathcal{N}_{\Delta\Lambda}\right)p^{\Lambda}-\tfrac{1}{2}\left(Im\mathcal{N}\right)^{-1|\Sigma\Lambda}q_{\Lambda}. \tag{3.3.1.4}$$

Now, by subtracting to the expression (3.3.1.4) its complex conjugate, one gets

$$p^{\Lambda}=2Re\left[i\overline{Z}L^{\Lambda}+ig^{i\bar{j}}\left(D_iZ\right)\overline{D}_{\bar{j}}\overline{L}^{\Lambda}\right]=-2Im\left[\overline{Z}L^{\Lambda}+g^{i\bar{j}}\left(D_iZ\right)\overline{D}_{\bar{j}}\overline{L}^{\Lambda}\right]. \tag{3.3.1.5}$$

On the other hand, by using the second of Ansätze (2.21), the contraction of both sides of Eq. (3.3.1.1) with $g^{i\bar{j}}D_jM_{\Sigma}$ analogously yields

$$g^{i\bar{j}}\left(D_iM_{\Sigma}\right)\overline{D}_{\bar{j}}\overline{Z}=q_{\Lambda}g^{i\bar{j}}\left(D_iM_{\Sigma}\right)\overline{D}_{\bar{j}}\overline{L}^{\Lambda}-p^{\Lambda}\mathcal{N}_{\Lambda\Delta}g^{i\bar{j}}\left(D_iM_{\Sigma}\right)\overline{D}_{\bar{j}}\overline{L}^{\Delta}=$$

$$=q_{\Lambda}g^{i\bar{j}}\overline{\mathcal{N}}_{\Sigma\Delta}\left(D_iL^{\Delta}\right)\overline{D}_{\bar{j}}\overline{L}^{\Lambda}-p^{\Lambda}\mathcal{N}_{\Lambda\Delta}g^{i\bar{j}}\overline{\mathcal{N}}_{\Sigma\Xi}\left(D_iL^{\Xi}\right)\overline{D}_{\bar{j}}\overline{L}^{\Delta}. \tag{3.3.1.6}$$

Once again, by exploiting the symmetry of $\mathcal{N}_{\Lambda\Sigma}$ and its inverse, recalling the first of the Ansätze (2.21), and using Eq. (3.3.1.3), Eq. (3.3.1.6) can be further elaborated as follows:

$$g^{i\bar{j}}\left(D_iM_{\Sigma}\right)\overline{D}_{\bar{j}}\overline{Z}=q_{\Lambda}\overline{\mathcal{N}}_{\Sigma\Delta}\left[-\tfrac{1}{2}\left(Im\mathcal{N}\right)^{-1|\Delta\Lambda}-\overline{L}^{\Delta}L^{\Lambda}\right]-p^{\Lambda}\mathcal{N}_{\Lambda\Delta}\overline{\mathcal{N}}_{\Sigma\Xi}\left[-\tfrac{1}{2}\left(Im\mathcal{N}\right)^{-1|\Xi\Delta}-\overline{L}^{\Xi}L^{\Delta}\right]=$$

$$=-\tfrac{1}{2}\left(Im\mathcal{N}\right)^{-1|\Delta\Lambda}\left(Re\mathcal{N}_{\Sigma\Delta}\right)q_{\Lambda}+\tfrac{i}{2}q_{\Sigma}-\overline{M}_{\Sigma}Z+\tfrac{1}{2}\left(Im\mathcal{N}\right)^{-1|\Xi\Delta}\left(Re\mathcal{N}_{\Sigma\Xi}\right)\left(Re\mathcal{N}_{\Lambda\Delta}\right)p^{\Lambda}+\tfrac{1}{2}\left(Im\mathcal{N}_{\Lambda\Sigma}\right)p^{\Lambda}. \tag{3.3.1.7}$$

Thence, by subtracting to the expression (3.3.1.7) its complex conjugate, one gets

$$q_{\Lambda}=2Re\left[i\overline{Z}M_{\Lambda}+ig^{i\bar{j}}\left(D_iZ\right)\overline{D}_{\bar{j}}\overline{M}_{\Lambda}\right]=-2Im\left[\overline{Z}M_{\Lambda}+g^{i\bar{j}}\left(D_iZ\right)\overline{D}_{\bar{j}}\overline{M}_{\Lambda}\right]. \tag{3.3.1.8}$$

By expressing the identities (3.3.1.5) and (3.3.1.8) in a vector $Sp\left(2n_V+2\right)$-covariant notation, one finally gets

$$\begin{pmatrix} p^{\Lambda} \\ q_{\Lambda} \end{pmatrix} = -2Im\left[\overline{Z}\begin{pmatrix} L^{\Lambda} \\ M_{\Lambda} \end{pmatrix}+g^{i\bar{j}}D_iZ\begin{pmatrix} \overline{D}_{\bar{j}}\overline{L}^{\Lambda} \\ \overline{D}_{\bar{j}}\overline{M}_{\Lambda} \end{pmatrix}\right], \tag{3.3.1.9}$$

or in compact form

$$Q^T=-2Im\left[\overline{Z}V+g^{i\bar{j}}\left(D_iZ\right)\overline{D}_{\bar{j}}\overline{V}\right], \tag{3.3.1.10}$$

where we recalled the definitions (1.1) and (2.4) of the $(2n_V+2)\times 1$ vectors Q^T and V, respectively.

It is worth pointing out that the vector identity (3.3.1.10) has been obtained only by using the properties of the SK geometry. The relations yielded by the identity (3.3.1.10) are $2n_V+2$ real ones, but they have been obtained by starting from an expression for $\overline{D}_{\bar{i}}\overline{Z}$, corresponding to n_V complex,

and therefore $2n_V$ real, degrees of freedom. The two redundant real degrees of freedom are encoded in the homogeneity (of degree 1) of the identity (3.3.1.10) under complex rescalings of the symplectic BH charge vector Q; indeed, by recalling the definition (2.2) it is immediate to check that the r.h.s. of identity (3.3.1.10) acquires an overall factor λ under a global rescaling of Q of the kind

$$Q \longrightarrow \lambda Q, \quad \lambda \in \mathbb{C}. \tag{3.3.1.11}$$

The summation of the expressions (3.3.1.4) and (3.3.1.7) with their complex conjugates respectively yields

$$(Im\mathcal{N})^{-1|\Delta\Lambda} (Re\mathcal{N}_{\Delta\Sigma}) p^{\Sigma} - (Im\mathcal{N})^{-1|\Lambda\Sigma} q_{\Sigma} = 2Re\left[\overline{Z}L^{\Lambda} + g^{i\overline{j}} (D_i Z) \overline{D}_{\overline{j}}\overline{L}^{\Lambda}\right]; \tag{3.3.1.12}$$

$$\begin{gathered}\left[Im\mathcal{N}_{\Lambda\Sigma} + (Im\mathcal{N})^{-1|\Xi\Delta} (Re\mathcal{N}_{\Lambda\Xi}) Re\mathcal{N}_{\Sigma\Delta}\right] p^{\Sigma} - (Im\mathcal{N})^{-1|\Delta\Sigma} (Re\mathcal{N}_{\Lambda\Delta}) q_{\Sigma} = \\ = 2Re\left[\overline{Z}M_{\Lambda} + g^{i\overline{j}} (D_i Z) \overline{D}_{\overline{j}}\overline{M}_{\Lambda}\right].\end{gathered} \tag{3.3.1.13}$$

In order to elaborate a shorthand notation for the obtained SKG identities (3.3.1.5), (3.3.1.8) and (3.3.1.12), (3.3.1.13), let us now reconsider the starting expressions (3.3.1.4) and (3.3.1.7), respectively reading

$$\left[\delta^{\Lambda}_{\Sigma} - i (Im\mathcal{N})^{-1|\Lambda\Delta} Re\mathcal{N}_{\Delta\Sigma}\right] p^{\Sigma} + i (Im\mathcal{N})^{-1|\Lambda\Sigma} q_{\Sigma} = -2i\overline{L}^{\Lambda} Z - 2ig^{i\overline{j}} \left(\overline{D}_{\overline{j}}\overline{Z}\right) D_i L^{\Lambda}; \tag{3.3.1.14}$$

$$\begin{gathered}-i\left[(Im\mathcal{N})^{-1|\Xi\Delta} (Re\mathcal{N}_{\Lambda\Xi}) Re\mathcal{N}_{\Sigma\Delta} + Im\mathcal{N}_{\Lambda\Sigma}\right] p^{\Sigma} + \left[\delta^{\Sigma}_{\Lambda} + i (Im\mathcal{N})^{-1|\Delta\Sigma} Re\mathcal{N}_{\Lambda\Delta}\right] q_{\Sigma} = \\ = -2i\overline{M}_{\Lambda} Z - 2ig^{i\overline{j}} \left(\overline{D}_{\overline{j}}\overline{Z}\right) D_i M_{\Lambda}.\end{gathered} \tag{3.3.1.15}$$

Thus, the identities (3.3.1.14) and (3.3.1.15) may be recast as the following fundamental $(2n_V + 2) \times 1$ vector identity, defining the geometric structure of SK manifolds [106, 9, 14, 17, 18, 23]:

$$Q^T - i\epsilon\mathcal{M}(\mathcal{N}) Q^T = -2i\overline{V}Z - 2ig^{i\overline{j}} \left(\overline{D}_{\overline{j}}\overline{Z}\right) D_i V. \tag{3.3.1.16}$$

The $(2n_V + 2) \times (2n_V + 2)$ real symmetric matrix $\mathcal{M}(\mathcal{N})$ is defined as [64, 3, 4]

$$\mathcal{M}(\mathcal{N}) = \mathcal{M}(Re\mathcal{N}, Im\mathcal{N}) \equiv \begin{pmatrix} Im\mathcal{N} + (Re\mathcal{N})(Im\mathcal{N})^{-1} Re\mathcal{N} & -(Re\mathcal{N})(Im\mathcal{N})^{-1} \\ -(Im\mathcal{N})^{-1} Re\mathcal{N} & (Im\mathcal{N})^{-1} \end{pmatrix}, \tag{3.3.1.17}$$

where $\mathcal{N}_{\Lambda\Sigma}$ is defined by Eq. (2.23). It is worth recalling that $\mathcal{M}(\mathcal{N})$ is symplectic with respect to the symplectic metric ϵ, *i.e.* it satisfies $((\mathcal{M}(\mathcal{N}))^T = \mathcal{M}(\mathcal{N}))$

$$\mathcal{M}(\mathcal{N}) \epsilon \mathcal{M}(\mathcal{N}) = \epsilon. \tag{3.3.1.18}$$

By using Eqs. (2.7), (2.24), (2.25) and (2.26), the identity (3.3.1.16) implies the following relations:

$$\left\{ \begin{array}{l} \left\langle V, Q^T - i\epsilon\mathcal{M}\left(\mathcal{N}\right)Q^T\right\rangle = -2Z; \\ \left\langle \overline{V}, Q^T - i\epsilon\mathcal{M}\left(\mathcal{N}\right)Q^T\right\rangle = 0; \\ \left\langle D_i V, Q^T - i\epsilon\mathcal{M}\left(\mathcal{N}\right)Q^T\right\rangle = 0; \\ \left\langle \overline{D}_{\overline{i}}\overline{V}, Q^T - i\epsilon\mathcal{M}\left(\mathcal{N}\right)Q^T\right\rangle = -2\overline{D}_{\overline{i}}\overline{Z}. \end{array} \right. \tag{3.3.1.19}$$

There are only $2n_V$ independent real relations out of the $4n_V + 4$ real ones yielded by the $2n_V + 2$ complex identities (3.3.1.16). Indeed, by taking the real and imaginary part of the SKG vector identity (3.3.1.16) one respectively obtains

$$Q^T = -2Re\left[iZ\overline{V} + ig^{i\overline{j}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)D_i V\right] = -2Im\left[\overline{Z}V + g^{i\overline{j}}\left(D_i Z\right)\overline{D}_{\overline{j}}\overline{V}\right]; \tag{3.3.1.20}$$

$$\epsilon\mathcal{M}\left(\mathcal{N}\right)Q^T = 2Im\left[iZ\overline{V} + ig^{i\overline{j}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)D_i V\right] = 2Re\left[\overline{Z}V + g^{i\overline{j}}\left(D_i Z\right)\overline{D}_{\overline{j}}\overline{V}\right]. \tag{3.3.1.21}$$

Consequently, the imaginary and real parts of the SKG vector identity (3.3.1.16) are *linearly dependent* one from the other, being related by the $(2n_V + 2) \times (2n_V + 2)$ real matrix

$$\epsilon\mathcal{M}\left(\mathcal{N}\right) = \begin{pmatrix} \left(Im\mathcal{N}\right)^{-1}Re\mathcal{N} & -\left(Im\mathcal{N}\right)^{-1} \\ Im\mathcal{N} + \left(Re\mathcal{N}\right)\left(Im\mathcal{N}\right)^{-1}Re\mathcal{N} & -\left(Re\mathcal{N}\right)\left(Im\mathcal{N}\right)^{-1} \end{pmatrix}. \tag{3.3.1.22}$$

Put another way, Eqs. (3.3.1.20) and (3.3.1.21) yield

$$Re\left[Z\overline{V} + g^{i\overline{j}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)D_i V\right] = \epsilon\mathcal{M}\left(\mathcal{N}\right)Im\left[Z\overline{V} + g^{i\overline{j}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)D_i V\right], \tag{3.3.1.23}$$

expressing the fact that the real and imaginary parts of the quantity $Z\overline{V} + g^{i\overline{j}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)D_i V$ are simply related through a finite *symplectic rotation* given by the matrix $\epsilon\mathcal{M}\left(\mathcal{N}\right)$ (see Eq. (3.4.1.10) further below), whose simplecticity directly follows from the symplectic nature of $\mathcal{M}\left(\mathcal{N}\right)$. Eq. (3.3.1.23) reduces the number of independent real relations implied by the identity (3.3.1.16) from $4n_V + 4$ to $2n_V + 2$. Two additional real degrees of freedom are scaled out by the complex rescaling (3.3.1.11).

This is clearly consistent with the fact that the $2n_V + 2$ complex identities (3.3.1.16) express nothing but a *change of basis* of the BH charge configurations, between the Kähler-invariant $1 \times (2n_V + 2)$ symplectic (magnetic/electric) basis vector Q defined by Eq. (1.1) and the complex, moduli-dependent $1 \times (n_V + 1)$ *matter charges* vector (with Kähler weights $(1, -1)$)

$$\mathcal{Z}\left(z, \overline{z}\right) \equiv \left(Z\left(z, \overline{z}\right), Z_i\left(z, \overline{z}\right)\right)_{i=1,\dots,n_V}. \tag{3.3.1.24}$$

It should be recalled that the BH charges are conserved due to the overall $\left(U(1)\right)^{n_V+1}$ gauge-invariance of the system under consideration, and Q and $\mathcal{Z}\left(z, \overline{z}\right)$ are two *equivalent* basis for them. Their very equivalence relations are given by the SKG identities (3.3.1.16) themselves. By its very definition (1.1), Q is *moduli-independent* (at least in a stationary, spherically symmetric and asymptotically flat extremal BH background, as it is the case being treated here), whereas Z is *moduli-dependent*, since it refers to the eigenstates of the $\mathcal{N} = 2$, $d = 4$ supergravity multiplet and of the n_V Maxwell vector supermultiplets.

3.3.2 *"New Attractor" Approach*

The evaluation of the (real part of the) fundamental SK geometrical identities (3.3.1.9) and (3.3.1.10) along the constraints determining the various classes of critical points of V_{BH} in $\mathcal{M}_{n_V}$ allows one to obtain a completely equivalent form of the AEs for extremal (static, spherically symmetric, asymptotically flat) BHs in $\mathcal{N}=2$, $d=4$ ungauged supergravity, which may be simpler in some cases (see also [26] for the treatment of an explicit case).

I) *Supersymmetric ($\frac{1}{2}$-BPS) critical points.* By evaluating the identities (3.3.1.9) and (3.3.1.10) along the constraints (3.1.1.1.1), one obtains

$$\left(\begin{array}{c} p^\Lambda \\ \\ q_\Lambda \end{array} \right) = -2Im \left[e^{K/2}\overline{Z} \left(\begin{array}{c} X^\Lambda \\ \\ F_\Lambda \end{array} \right) \right]_{\frac{1}{2}-BPS}, \tag{3.3.2.1}$$

or in compact form

$$Q^T = -2Im \left[e^{K/2}\overline{Z}\Pi \right]_{\frac{1}{2}-BPS}.$$

Eqs. (3.3.2.1) and (3.3.2.2) are equivalent, purely algebraic forms of the $\frac{1}{2}$-BPS extremal BH AEs, given by the (partly differential) conditions (3.1.1.1.1). By inserting as input the BH charge configuration $Q \equiv \left(p^\Lambda, q_\Lambda\right)$ and the covariantly holomorphic sections L^Λ and M_Λ of the $U(1)$-bundle over $\mathcal{M}_{n_V}$, Eqs. (3.3.2.1) and (3.3.2.2) give as output (if any) the purely charge-dependent $\frac{1}{2}$-BPS critical points $\left(z^i_{\frac{1}{2}-BPS}(p,q), \overline{z}^{\overline{i}}_{\frac{1}{2}-BPS}(p,q)\right)$ of V_{BH}.

By looking at Eqs. (3.3.2.1) and (3.3.2.2), it is easy to realize that $\frac{1}{2}$-BPS critical points of V_{BH} with $Z=0$ (which are *degenerate*, yielding $V_{BH,\frac{1}{2}-BPS}=0$) correspond to the trivial case of *all* vanishing magnetic and electric BH charges. This means that (static, spherically symmetric, asymptotically flat) extremal BHs with $\frac{1}{2}$-BPS attractor horizon scalar configurations with $Z=0$ (*i.e.* with no central extension of the $\mathcal{N}=2$, $d=4$ horizon supersymmetry algebra) cannot be described by the classical extremal BH Attractor Mechanism encoded by Eqs. (3.3.2.1) and (3.3.2.2). They are a particular case of the so-called *"small"* extremal BHs, which are *classically degenerate*, acquiring a non-vanishing, finite horizon area and entropy only taking into account quantum/higher-derivative corrections.

It is worth pointing out that Eqs. (3.3.2.1) and (3.3.2.2) are purely algebraic ones, whereas Eqs. (3.1.1.1.1) are (partly) differential, thus, in general, more complicated to be solved. Consequently, at least in the $\frac{1}{2}$-BPS case, the *"new attractor" approach* is simpler of the *"criticality conditions" approach* to the search of critical points of V_{BH}.

II) *Non-BPS $Z \neq 0$ critical points.* By evaluating the identities (3.3.1.9) and (3.3.1.10) along the

constraints (3.1.1.2.1) and (3.1.1.2.2), one obtains

$$\left(\begin{array}{c} p^{\Lambda} \\ \\ q_{\Lambda} \end{array}\right) =$$

$$= 2Im\left\{e^{K/2}\left[Z\left(\begin{array}{c} \overline{X}^{\Lambda} \\ \\ \overline{F}_{\Lambda} \end{array}\right) + \tfrac{i}{2}\tfrac{\overline{Z}}{|Z|^2}\overline{C}_{\overline{i}\overline{j}\overline{k}}g^{i\overline{i}}g^{j\overline{j}}g^{k\overline{k}}\left(D_j Z\right)\left(D_k Z\right)\left(\begin{array}{c} D_i X^{\Lambda} \\ \\ D_i F_{\Lambda} \end{array}\right)\right]\right\}_{non-BPS,Z\neq 0}, \tag{3.3.2.2}$$

or in compact form

$$Q^T = 2Im\left\{e^{K/2}\left[Z\overline{\Pi} + \frac{i}{2}\frac{\overline{Z}}{|Z|^2}\overline{C}_{\overline{i}\overline{j}\overline{k}}g^{i\overline{i}}g^{j\overline{j}}g^{k\overline{k}}\left(D_j Z\right)\left(D_k Z\right)D_i\Pi\right]\right\}_{non-BPS,Z\neq 0}. \tag{3.3.2.3}$$

Eqs. (3.3.2.2) and (3.3.2.3) are equivalent forms of the non-BPS $Z \neq 0$ extremal BH AEs, given by the (partly differential) conditions (3.1.1.2.1) and (3.1.1.2.2). By inserting as input the BH charge configuration Q, the covariantly holomorphic sections L^{Λ} and M_{Λ}, the Kähler potential K (and consequently the contravariant metric tensor $g^{i\overline{j}}$) and the completely symmetric, covariantly holomorphic rank-3 tensor C_{ijk}, Eqs. (3.3.2.2) and (3.3.2.3) give as output (if any) the purely charge-dependent non-BPS $Z \neq 0$ critical points $\left(z^i_{non-BPS,Z\neq 0}(p,q), \overline{z}^{\overline{i}}_{non-BPS,Z\neq 0}(p,q)\right)$ of V_{BH}. Notice that, differently from Eqs. (3.3.2.1) and (3.3.2.2), Eqs. (3.3.2.2) and (3.3.2.3) are not purely algebraic. Thus, in the non-BPS $Z \neq 0$ case the (computational) simplification in the search of critical points of V_{BH} obtained by exploiting the *"new attractor" approach* rather than the *"criticality conditions" approach* is model-dependent.

It is interesting to point out that, as it is evident by looking for instance at Eq. (3.3.2.3), at the non-BPS $Z \neq 0$ critical points of V_{BH} the coefficients of $\overline{\Pi}$ and $D_i\Pi$ in the AEs have the same holomorphicity in the central charge Z, *i.e.* they can be expressed only in terms of Z and D_iZ, without using $\overline{Z}$ and $\overline{D}_{\overline{i}}\overline{Z}$. Such a fact does not happen in a generic point of $\mathcal{M}_{n_V}$, as it is seen from the global identity (3.3.1.10). As it is evident, the price to be paid in order to obtain the same holomorphicity in Z at the non-BPS $Z \neq 0$ critical points of V_{BH} is the fact that the coefficient of $D_i\Pi$ is not linear in some covariant derivative of Z any more, also explicitly depending on the rank-3 covariantly antiholomorphic tensor $\overline{C}_{\overline{i}\overline{j}\overline{k}}$.

III) *Non-BPS $Z = 0$ critical points.* By evaluating the identities (3.3.1.9) and (3.3.1.10) along the constraints (3.1.1.3.1) and (3.1.1.3.2), one obtains

$$\left(\begin{array}{c} p^{\Lambda} \\ \\ q_{\Lambda} \end{array}\right) = 2Im\left\{e^{K/2}\left[g^{i\overline{j}}\left(\overline{\partial}_{\overline{j}}\overline{Z}\right)\left(\begin{array}{c} D_i X^{\Lambda} \\ \\ D_i F_{\Lambda} \end{array}\right)\right]\right\}_{non-BPS,Z=0}, \tag{3.3.2.4}$$

or in compact form

$$Q^T = 2Im\left\{e^{K/2}\left[g^{i\overline{j}}\left(\overline{\partial}_{\overline{j}}\overline{Z}\right)D_i\Pi\right]\right\}_{non-BPS,Z=0}. \tag{3.3.2.5}$$

Eqs. (3.3.2.4) and (3.3.2.5) are equivalent forms of the non-BPS $Z=0$ extremal BH AEs, given by the (partly differential) conditions (3.1.1.3.1) and (3.1.1.3.2). By inserting as input the BH charge configuration Q, the covariantly holomorphic sections L^Λ and M_Λ, and the Kähler potential K (and consequently the contravariant metric tensor $g^{i\bar{j}}$), Eqs. (3.3.2.4) and (3.3.2.5) give as output (if any) the purely charge-dependent non-BPS $Z=0$ critical points $\left(z^i_{non-BPS,Z=0}(p,q), \bar{z}^{\bar{i}}_{non-BPS,Z=0}(p,q)\right)$ of V_{BH}. Differently from Eqs. (3.3.2.1) and (3.3.2.2), and similarly to Eqs. (3.3.2.2) and (3.3.2.3), Eqs. (3.3.2.4) and (3.3.2.5) are not purely algebraic. Thus, in the non-BPS $Z=0$ case the (computational) simplification in the search of critical points of V_{BH} obtained by exploiting the *"new attractor" approach* rather than the *"criticality conditions" approach* is model-dependent.

3.4 Type IIB Superstrings on CY_3

3.4.1 Hodge Decomposition of $\mathcal{H}_3$

We consider Type IIB superstring theory compactified on a Calabi-Yau threefold (CY_3) [107, 108, 64, 66], determining an effective $\mathcal{N}=2$, $d=4$ ungauged supergravity with a number n_V of Abelian vector multiplets. Within such a framework, the CY_3 has a complex structure (CS) moduli space (of complex dimension $n_V = h_{2,1} \equiv dim\left(H^{2,1}(CY_3)\right)$, where $H^{2,1}$ is the $(2,1)$-cohomology group of the considered manifold), which is a special Kähler (SK) manifold.

We introduce a[10] b_3-dim. real (manifestly symplectic-covariant) basis of the third real[11] cohomology $H^3(CY_3,\mathbb{R})$, given by the set of real 3-forms $\left\{\alpha_\Lambda, \beta^\Lambda\right\}$ ($\Lambda = 0,1,...,h_{2,1}$ throughout) satisfying[12]

$$\int_{CY_3} \alpha_\Lambda \wedge \alpha_\Sigma = 0, \quad \int_{CY_3} \beta^\Lambda \wedge \beta^\Sigma = 0, \quad \int_{CY_3} \alpha_\Lambda \wedge \beta^\Sigma = \delta^\Sigma_\Lambda. \tag{3.4.1.1}$$

By Poincarè-duality on CY_3, we may correspondingly introduce the b_3-dim. real (manifestly symplectic-covariant) basis of the third real homology $H_3(CY_3,\mathbb{R})$, given by the set of real 3-cycles $\left\{A^\Lambda, B_\Lambda\right\}$ satisfying

$$\int_{A^\Lambda} \alpha_\Sigma = \delta^\Lambda_\Sigma, \quad \int_{A^\Lambda} \beta^\Sigma = 0, \quad \int_{B_\Lambda} \alpha_\Sigma = 0, \quad \int_{B_\Lambda} \beta^\Sigma = -\delta^\Sigma_\Lambda. \tag{3.4.1.2}$$

The CY_3 is endowed with a (*nowhere-vanishing*) holomorphic 3-form

$$\Omega_3(z) \equiv X^\Lambda(z)\,\alpha_\Lambda - F_\Lambda(z)\,\beta^\Lambda \in H^{3,0}(CY_3), \tag{3.4.1.3}$$

where "z" denotes the functional dependence on the CS moduli $\left\{z^i, \bar{z}^{\bar{i}}\right\}$ ($i = 1,...,h_{2,1}$ throughout), and $\left\{X^\Lambda, F_\Lambda\right\}$ stands for the basis of symplectic holomorphic fundamental periods of Ω_3 around the 3-cycles $\left\{A^\Lambda, B_\Lambda\right\}$, respectively:

$$X^\Lambda(z) \equiv \int_{A^\Lambda} \Omega_3(z), \quad F_\Lambda(z) \equiv \int_{B_\Lambda} \Omega_3(z). \tag{3.4.1.4}$$

Ω_3, as well as its fundamental periods, has Kähler weights $(2,0)$:

$$\begin{aligned} &D_i\Omega_3 = \partial_i\Omega_3 + (\partial_i K)\,\Omega_3, \\ &\overline{D}_{\bar{i}}\Omega_3 = \overline{\partial}_{\bar{i}}\Omega_3 = 0, \end{aligned} \tag{3.4.1.5}$$

[10] $b_3 = 2h_{2,1}+2$ is the so-called third Betti number of the CY_3.

[11] In the strict quantum regime, one should consider the third *integer* cohomology $H^3(CY_3,\mathbb{Z})$. The present (semi)classical treatment deal with the *large charges limit* and thus consistently consider real, unquantized, rather than integer, quantized quantities.

[12] Recall that the $\wedge$ ("wedge") product among odd-forms is odd, whereas the one among even-forms (and among odd- and even-forms) is even.

where K is the real Kähler potential in the $h_{2,1}$-dim. SK CS moduli space of CY_3.

Type IIB compactified on CY_3 is characterized by a real 5-form

$$\mathcal{Z} \equiv \mathcal{F}^{\Lambda}\alpha_{\Lambda} - \mathcal{G}_{\Lambda}\beta^{\Lambda}, \tag{3.4.1.6}$$

where $\mathcal{F}^{\Lambda}$ is the space-time 2-form given by the Abelian field-strengths ($\Lambda = 0$ pertains to the graviphoton, whereas $\Lambda = i$ corresponds to the Maxwell vector supermultiplets), and $\mathcal{G}_{\Lambda}$ is the corresponding "dual" space-time 2-form, in the sense of Legendre transform:

$$\mathcal{G}_{\Lambda} \equiv \frac{\delta \mathcal{L}}{\delta \mathcal{F}^{\Lambda}} = \left(ReN_{\Lambda\Sigma}\right)\mathcal{F}^{\Sigma} + \frac{1}{2}\left(ImN_{\Lambda\Sigma}\right)^{*}\mathcal{F}^{\Sigma}. \tag{3.4.1.7}$$

${}^{*}\mathcal{F}^{\Sigma}$ denotes the Hodge $*$-dual of $\mathcal{F}^{\Lambda}$, defined in components as follows (the space-time indices μ, ν run $0,1,2,3$ throughout):

$${}^{*}\mathcal{F}^{\Lambda}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\Lambda|\rho\sigma} = \frac{1}{2}G^{\rho\lambda}G^{\sigma\tau}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\Lambda}_{\lambda\tau}, \tag{3.4.1.8}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the $d = 4$ completely antisymmetric Ricci-Levi-Civita tensor, and $G^{\mu\nu}$ is the $d = 4$ space-time completely contravariant metric tensor. $\mathcal{L}$ stands for the (bosonic sector of the) $\mathcal{N} = 2$, $d = 4$ ungauged supergravity Lagrangian density:

$$\mathcal{L} = -\frac{R}{2} + g_{i\bar{j}}\left(\partial_{\mu}z^{i}\right)\left(\partial_{\nu}\bar{z}^{\bar{j}}\right)G^{\mu\nu} + \frac{1}{4}\left(ImN_{\Lambda\Sigma}\right)G^{\mu\lambda}G^{\nu\rho}\mathcal{F}^{\Lambda}_{\mu\nu}\mathcal{F}^{\Lambda}_{\lambda\rho} + \frac{1}{4}\left(ReN_{\Lambda\Sigma}\right)G^{\mu\lambda}G^{\nu\rho}\mathcal{F}^{\Lambda}_{\mu\nu}{}^{*}\mathcal{F}^{\Lambda}_{\lambda\rho}. \tag{3.4.1.9}$$

The Hodge $*$-duality acts as a symplectic $Sp\left(2h_{1,2}+2,\mathbb{R}\right)$ rotation on the basis $\left\{\alpha_{\Lambda}, \beta^{\Lambda}\right\}$:

$$\begin{pmatrix} {}^{*}\alpha_{\Lambda} \\ {}^{*}\beta^{\Lambda} \end{pmatrix} = \mathcal{S}\begin{pmatrix} \alpha_{\Lambda} \\ \beta^{\Lambda} \end{pmatrix}, \quad \mathcal{S} \equiv -\epsilon\mathcal{M}\left(\mathcal{N}\right), \quad \mathcal{S}^{T}\epsilon\mathcal{S} = \epsilon. \tag{3.4.1.10}$$

where ϵ is the $(2h_{1,2}+2)$-dim. symplectic metric defined in Eq. (2.3), $\mathcal{M}\left(\mathcal{N}\right)$ is the real, symplectic matrix defined by Eq. (3.3.1.17). Notice that $\mathcal{S}$ is nothing but the opposite of the matrix given by Eq. (3.3.1.22). It can be shown that $\mathcal{Z}$ is Hodge $*$-self-dual:

$${}^{*}\mathcal{Z} = \mathcal{Z}. \tag{3.4.1.11}$$

Whenever the relevant integrations over internal manifold CY_3 and over space-time make sense, manifestly symplectic-covariant magnetic and electric charges can be introduced as the asymptotical "space-dressings" of a suitable contraction of $\mathcal{Z}$ over the symplectic 3-cycles of CY_3, *i.e.* as the asymptotical fluxes of the space-time 2-forms corresponding to the components of $\mathcal{Z}$ along the symplectic basis $\left\{A^{\Lambda}, B_{\Lambda}\right\}$ of $H_3\left(CY_3,\mathbb{R}\right)$, respectively:

$$\begin{aligned} p^{\Lambda} &\equiv \tfrac{1}{4\pi}\int_{A^{\Lambda}\times S^{2}_{\infty}}\mathcal{Z} = \tfrac{1}{4\pi}\int_{A^{\Lambda}\times S^{2}_{\infty}}\left(\mathcal{F}^{\Sigma}\alpha_{\Sigma} - \mathcal{G}_{\Sigma}\beta^{\Sigma}\right) = \tfrac{1}{4\pi}\int_{S^{2}_{\infty}}\mathcal{F}^{\Lambda}; \\ q_{\Lambda} &\equiv \tfrac{1}{4\pi}\int_{B_{\Lambda}\times S^{2}_{\infty}}\mathcal{Z} = \tfrac{1}{4\pi}\int_{B_{\Lambda}\times S^{2}_{\infty}}\left(\mathcal{F}^{\Sigma}\alpha_{\Sigma} - \mathcal{G}_{\Sigma}\beta^{\Sigma}\right) = \tfrac{1}{4\pi}\int_{S^{2}_{\infty}}\mathcal{G}^{\Lambda}, \end{aligned} \tag{3.4.1.12}$$

where S^{2}_{∞} denotes the 2-sphere at spatial infinity[13].

[13]Consistently with (a proper subset of) the solutions of $\mathcal{N} = 2$, $d = 4$ ungauged supergravity, the space-time metric is assumed to be static, spherically symmetric, and asymptotically flat. In such a framework, "spatial infinity" corresponds to $r \to \infty$, where r is the radial coordinate.

$\{p^\Lambda, q_\Lambda\}$ can be seen as the components (along the real symplectic basis $\{\alpha_\Lambda, \beta^\Lambda\}$ of $H^3(CY_3, \mathbb{R})$) of the real flux 3-form $\mathcal{H}_3$, defined as the asymptotical "space-dressing" of $\mathcal{Z}$:

$$\mathcal{H}_3 \equiv \frac{1}{4\pi}\int_{S^2_\infty} \mathcal{Z} = p^\Lambda \alpha_\Lambda - q_\Lambda \beta^\Lambda \in H^3(CY_3, \mathbb{R}). \quad (3.4.1.13)$$

$\{p^\Lambda, q_\Lambda\}$ are the physical charges, and they are conserved, due to the overall $(U(1))^{h_{2,1}+1}$ gauge symmetry of the considered framework. They respectively are the magnetic and electric charges of the $(U(1))^{h_{2,1}+1}$ gauge group of the (symplectic) real parameterization of $H^3(CY_3, \mathbb{R})$, which however is not the only possible one.

Indeed, in general the third real cohomology $H^3(CY_3, \mathbb{R})$ can be Hodge-decomposed along the third Dalbeault cohomogy of CY_3 as follows:

$$H^3(CY_3, \mathbb{R}) = H^{3,0}(CY_3) \oplus_s H^{2,1}(CY_3) \oplus_s H^{1,2}(CY_3) \oplus_s H^{0,3}(CY_3), \quad (3.4.1.14)$$

corresponding to perform a change of basis from the symplectic real basis to the Dalbeault basis:

$$\{\alpha_\Lambda, \beta^\Lambda\} \longrightarrow \{\Omega_3, D_i\Omega_3, \overline{D}_{\bar{i}}\overline{\Omega}_3, \overline{\Omega}_3\}. \quad (3.4.1.15)$$

The subscript "s" in in Eq. (3.4.1.14) stands for the semidirect cohomological sum, due to the fact that (some of the) cohomologies in the r.h.s. of the Hodge decomposition (3.4.1.14) have non-vanishing intersections. Indeed, as it can be checked by recalling Eqs. (2.24), (2.25) and (3.4.1.1), the following results hold:

$$\int_{CY_3} \Omega_3 \wedge \Omega_3 = 0, \quad \int_{CY_3} \Omega_3 \wedge D_i\Omega_3 = 0, \quad \int_{CY_3} \Omega_3 \wedge \overline{D}_{\bar{i}}\overline{\Omega}_3 = 0,$$

$$\int_{CY_3} \Omega_3 \wedge \overline{\Omega}_3 = -ie^{-K} \Leftrightarrow K = -ln\left(i\int_{CY_3} \Omega_3 \wedge \overline{\Omega}_3\right),$$

$$\int_{CY_3} (D_i\Omega_3) \wedge D_j\Omega_3 = 0,$$

$$\begin{gathered}\int_{CY_3} (D_i\Omega_3) \wedge \overline{D}_{\bar{j}}\overline{\Omega}_3 = \left[\overline{\partial}_{\bar{j}}\partial_i ln\left(i\int_{CY_3} \Omega_3 \wedge \overline{\Omega}_3\right)\right]\int_{CY_3} \Omega_3 \wedge \overline{\Omega}_3 = -\overline{\partial}_{\bar{j}}\partial_i K \int_{CY_3} \Omega_3 \wedge \overline{\Omega}_3 = ie^{-K} g_{i\bar{j}} \\ \Updownarrow \\ g_{i\bar{j}} = -\overline{\partial}_{\bar{j}}\partial_i ln\left(i\int_{CY_3} \Omega_3 \wedge \overline{\Omega}_3\right) = -\frac{\int_{CY_3} (D_i\Omega_3) \wedge \overline{D}_{\bar{j}}\overline{\Omega}_3}{\int_{CY_3} \Omega_3 \wedge \overline{\Omega}_3}.\end{gathered} \quad (3.4.1.16)$$

In particular, the second line of Eq. (3.4.1.16) allows one to write the covariant derivatives of Ω_3 (which are the basis of $H^{2,1}(CY_3)$) as follows:

$$D_i\Omega_3 = \left(\partial_i - \frac{\int_{CY_3} (\partial_i\Omega_3) \wedge \overline{\Omega}_3}{\int_{CY_3} \Omega_3 \wedge \overline{\Omega}_3}\right)\Omega_3. \quad (3.4.1.17)$$

It is worth pointing out that the $2h_{2,1}+2$ 3-forms $\{\Omega_3, D_i\Omega_3, \overline{D}_{\bar{i}}\overline{\Omega}_3, \overline{\Omega}_3\}_{i=1,\dots,h_{2,1}}$ are *all* the possible $((2,0)$ and $(0,2))$-Kähler-weighted independent 3-forms which can be defined on CY_3 in the considered framework. This is due to the two fundamental relations

$$\overline{D}_{\bar{j}} D_i\Omega_3 = g_{i\bar{j}}\Omega_3; \quad (3.4.1.18)$$

$$D_i D_j\Omega_3 = iC_{ijk}g^{k\bar{l}}\overline{D}_{\bar{l}}\overline{\Omega}_3 = D_{(i}D_{j)}\Omega_3, \quad (3.4.1.19)$$

which are the translation, in the language of forms on CY_3, of the third and second of Eqs. (2.17), respectively. Notice that the third of Eqs. (2.17) and Eq. (3.4.1.18) hold in a generic Kähler framework, whereas the second of Eqs. (2.17) and Eq. (3.4.1.19) in general hold only in SK geometry. Due to Eq. (3.4.1.19), the completely symmetric, covariantly holomorphic tensor C_{ijk} of SK geometry can be obtained by intersecting the elements of the basis of $H^{2,1}(CY_3)$ with their covariant derivatives (and normalizing with respect to the intersection of $H^{3,0}(CY_3)$ and $H^{0,3}(CY_3)$):

$$C_{ijk} = C_{(ijk)} = -i\frac{\int_{CY_3}(D_iD_j\Omega_3)\wedge D_k\Omega_3}{\int_{CY_3}\Omega_3\wedge\overline{\Omega}_3} = e^K\int_{CY_3}(D_iD_j\Omega_3)\wedge D_k\Omega_3. \tag{3.4.1.20}$$

According to the Hodge-decomposition (3.4.1.14) implemented through the change of basis (3.4.1.15), the charges undergo the following change of basis:

$$\left\{p^\Lambda, q_\Lambda\right\} \longrightarrow \left\{Z^{3,0}(z,p,q), Z_i^{2,1}(z,\overline{z},p,q), Z_{\overline{i}}^{1,2}(z,\overline{z},p,q), Z^{0,3}(\overline{z},p,q)\right\}, \tag{3.4.1.21}$$

where the complex, (CS) moduli-dependent quantities on the r.h.s. are defined as follows:

$$Z^{3,0}(z;p,q) \equiv \int_{CY_3}\mathcal{H}_3\wedge\Omega_3(z) = \frac{1}{4\pi}\int_{CY_3}\left(\int_{S^2_\infty}\mathcal{Z}\right)\wedge\Omega_3(z) = X^\Lambda(z)q_\Lambda - F_\Lambda(z)p^\Lambda = W(z;p,q); \tag{3.4.1.22}$$

$$\begin{aligned}Z_i^{2,1}(z,\overline{z};p,q) &\equiv \int_{CY_3}\mathcal{H}_3\wedge(D_i\Omega_3)(z,\overline{z}) = \int_{CY_3}\left(\int_{S^2_\infty}\mathcal{Z}\right)\wedge(D_i\Omega_3)(z,\overline{z}) = \\ &= \left(D_iX^\Lambda\right)(z,\overline{z})q_\Lambda - (D_iF_\Lambda)(z,\overline{z})p^\Lambda = (D_iW)(z,\overline{z};p,q);\end{aligned} \tag{3.4.1.23}$$

$$Z_{\overline{i}}^{1,2}(z,\overline{z};p,q) \equiv \overline{Z_i^{2,1}(z,\overline{z};p,q)}; \tag{3.4.1.24}$$

$$Z^{0,3}(\overline{z};p,q) \equiv \overline{Z^{3,0}(z;p,q)}. \tag{3.4.1.25}$$

As it can be seen, $Z^{3,0}(z;p,q)$ and $Z_i^{2,1}(z,\overline{z};p,q)$ are respectively nothing but the $\mathcal{N}=2$, $d=4$ holomorphic central charge function $W(z;p,q)$, also named $\mathcal{N}=2$ superpotential (see Eq. (2.2) and comments below), and its covariant derivatives, introduced *à la Gukov-Vafa-Witten* (GVW) [83] also in the considered $\mathcal{N}=2$, $d=4$ framework. In other words, Eq. (3.4.1.22) defines the holomorphic central extension of the $\mathcal{N}=2$, $d=4$ local supersymmetry algebra, whereas Eq. (3.4.1.22) defines in a geometrical way the charges of the other field strength vectors, orthogonal to the graviphoton. $\left\{Z^{3,0}, Z_i^{2,1}, Z_{\overline{i}}^{1,2}, Z^{0,3}\right\}$ correspond to electric and magnetic charges of the $(U(1))^{h_{2,1}+1}$ gauge group of the complex Dalbeault parameterization of $H^3(CY_3,\mathbb{R})$. Their dependence on moduli can be understood by taking into account that they refer to the supermultiplet eigenstates, which are moduli-dependent (as already pointed out below Eq. (3.3.1.23)). They satisfy the following model-independent sum rules [3]:

$$\left(\left|Z^{3,0}\right|^2 + g^{i\overline{j}}Z_i^{2,1}Z_{\overline{j}}^{1,2}\right)e^K = -\frac{1}{2}\left(p^\Lambda, q_\Lambda\right)\mathcal{M}(\mathcal{N})\begin{pmatrix}p^\Sigma\\ q_\Sigma\end{pmatrix} = I_1(z,\overline{z};p,q) = V_{BH}(z,\overline{z};p,q) \geqslant 0; \tag{3.4.1.26}$$

$$\left(\left|Z^{3,0}\right|^2 - g^{i\overline{j}}Z_i^{2,1}Z_{\overline{j}}^{1,2}\right)e^K = -\frac{1}{2}\left(p^\Lambda, q_\Lambda\right)\mathcal{M}(\mathcal{F})\begin{pmatrix}p^\Sigma\\ q_\Sigma\end{pmatrix} = I_2(z,\overline{z};p,q) \gtreqless 0, \tag{3.4.1.27}$$

where $\mathcal{M}(\mathcal{N})$ is the real, symplectic matrix defined by Eq. (3.3.1.17), $\mathcal{F} \equiv F_{\Lambda\Sigma} = \partial_\Sigma F_\Lambda$, $\mathcal{M}(\mathcal{F}) = \mathcal{M}(\mathcal{N}\rightarrow\mathcal{F})$. I_1 and I_2 are the first and second lowest-order (quadratic in charges) invariants of SK

geometry. As far as the metric $g_{i\bar{j}}$ of the SK CS moduli space is regular, I_1 has positive signature and it is nothing but the "BH effective potential" V_{BH}, whereas I_2 has signature $(1, h_{2,1})$. Since the considered extremal BH background is static (and spherically symmetric), the undressed charges p^Λ and q_Λ are conserved in time, and so are the dressed charges $\left\{Z^{3,0}, Z_i^{2,1}, Z_{\bar{i}}^{1,2}, Z^{0,3}\right\}$ (which however, through their dependence on scalars, do depend on radial coordinate).

The real, Kähler gauge-invariant 3-form $\mathcal{H}_3$ can be thus Hodge-decomposed as follows ($\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \mathbb{C}$)

$$\begin{array}{l}
\mathcal{H}_3 = e^K \left[\begin{array}{l} \gamma_1 \left(\int_{CY_3} \mathcal{H}_3 \wedge \Omega_3 \right) \overline{\Omega}_3 + \gamma_2 g^{i\bar{j}} \left(\int_{CY_3} \mathcal{H}_3 \wedge (D_i\Omega_3) \right) \overline{D}_{\bar{j}} \overline{\Omega}_3 + \\ \\ + \gamma_3 g^{j\bar{i}} \left(\int_{CY_3} \mathcal{H}_3 \wedge \overline{D}_{\bar{i}} \overline{\Omega}_3 \right) D_j \Omega_3 + \gamma_4 \left(\int_{CY_3} \mathcal{H}_3 \wedge \overline{\Omega}_3 \right) \Omega_3 \end{array} \right] = \\ \\
= e^K \left[\begin{array}{l} \gamma_1 W \overline{\Omega}_3 + \gamma_2 g^{i\bar{j}} (D_i W) \overline{D}_{\bar{j}} \overline{\Omega}_3 + \\ \\ + \gamma_3 g^{j\bar{i}} \left(\overline{D}_{\bar{i}} \overline{W} \right) D_j \Omega_3 + \gamma_4 \overline{W} \Omega_3 \end{array} \right],
\end{array} \tag{3.4.1.28}$$

where Eqs. (3.4.1.22)-(3.4.1.25) were used. The r.h.s. of the Hodge-decomposition (3.4.1.28) is the most general Kähler gauge-invariant combination of *all* the possible ((2,0) and (0,2))-Kähler-weighted independent 3-forms for Type IIB on CY_3s. The overall factor e^K (with Kähler weights $(-2,-2)$) is necessary to make the r.h.s. of the identity (3.4.1.28) Kähler gauge-invariant. The reality condition $\overline{\mathcal{H}}_3 = \mathcal{H}_3$ implies $\gamma_3 = \overline{\gamma_2}$ and $\gamma_4 = \overline{\gamma_1}$. The complex coefficients γ_1 and γ_2 can be determined by computing $\int_{CY_3} \mathcal{H}_3 \wedge \Omega_3$ and $\int_{CY_3} \mathcal{H}_3 \wedge D_l \Omega_3$, using the identity (3.4.1.28) and recalling Eqs. (3.4.1.22)-(3.4.1.25) and the intersections (3.4.1.16). By doing so, one obtains:

$$\begin{array}{l}
W = \int_{CY_3} \mathcal{H}_3 \wedge \Omega_3 = e^K \gamma_1 W \int_{CY_3} \overline{\Omega}_3 \wedge \Omega_3 = i\gamma_1 W \Leftrightarrow \gamma_1 = -i; \\ \\
D_l W = \int_{CY_3} \mathcal{H}_3 \wedge D_l \Omega_3 = e^K \gamma_2 g^{i\bar{j}} (D_i W) \int_{CY_3} \overline{D}_{\bar{j}} \overline{\Omega}_3 \wedge D_l \Omega_3 = -i\gamma_2 D_l W \Leftrightarrow \gamma_2 = i.
\end{array} \tag{3.4.1.29}$$

Thus, the complete Hodge-decomposition of the real flux 3-form $\mathcal{H}_3$ of Type IIB on CY_3s reads

$$\begin{array}{l}
\mathcal{H}_3 = -ie^K \left[W \overline{\Omega}_3 - g^{i\bar{j}} (D_i W) \overline{D}_{\bar{j}} \overline{\Omega}_3 + g^{j\bar{i}} \left(\overline{D}_{\bar{i}} \overline{W} \right) D_j \Omega_3 - \overline{W} \Omega_3 \right] = \\ \\
= -2Im \left[\overline{Z} \hat{\Omega}_3 - g^{j\bar{i}} \left(\overline{D}_{\bar{i}} \overline{Z} \right) D_j \hat{\Omega}_3 \right],
\end{array} \tag{3.4.1.30}$$

where in the second line we recalled the definition of the $\mathcal{N} = 2$, $d = 4$ covariantly holomorphic central charge function $Z(z, \overline{z}, p, q)$ (with Kähler weights $(1, -1)$) given by Eq. (2.2) (see also Eq. (2.16)), and

introduced the covariantly holomorphic $(3,0)$-form $\hat{\Omega}_3$ (with Kähler weights $(1,-1)$) on CY_3:

$$\begin{aligned} &Z(z,\overline{z};p,q) \equiv e^{\frac{K(z,\overline{z})}{2}} W(z;p,q), \\ &D_i Z = e^{\frac{K}{2}} D_i W,\ \overline{D}_{\overline{i}} Z = 0; \end{aligned} \tag{3.4.1.31}$$

$$\begin{aligned} &\hat{\Omega}_3(z,\overline{z}) \equiv \frac{\Omega_3}{\sqrt{i\int_{CY_3}\Omega_3\wedge\overline{\Omega}_3}} = e^{\frac{K(z,\overline{z})}{2}}\Omega_3(z), \\ &D_i\hat{\Omega}_3 = e^{\frac{K}{2}} D_i\Omega_3,\ \overline{D}_{\overline{i}}\hat{\Omega}_3 = 0. \end{aligned} \tag{3.4.1.32}$$

Let us now compare the Hodge-decomposition identity (3.4.1.30) with the real part (3.3.1.9) and (3.3.1.10) of the SK geometrical identities (3.3.1.16). It is immediate to realize that the identity (3.4.1.30) is nothing but the translation, in the language of forms of Type IIB on CY_3 (*i.e.* in a particular stringy framework) of the identity (3.3.1.9)-(3.3.1.10), which is the real part of the fundamental identities (3.3.1.16), holding for any SK geometry, irrespectively of its microscopic/stringy origin.

3.4.2 *"New Attractor" Approach*

The evaluation of the Hodge-decomposition identity (3.4.1.30) along the constraints determining the various classes of critical points of V_{BH} in $\mathcal{M}_{n_V}$ (which in the considered stringy framework is nothing but the CS moduli space of CY_3) allows one to obtain a completely equivalent form of the AEs for extremal (static, spherically symmetric, asymptotically flat) BHs in the particular framework in which $\mathcal{N}=2$, $d=4$ ungauged supergravity is obtained by compactifying Type IIB on CY_3. As already pointed out in the treatment at macroscopic level, in some cases such equivalent forms of AEs may be simpler to solve than the AEs obtained by exploiting the *"criticality conditions" approach* (see Sect. 3.1).

I) *Supersymmetric ($\frac{1}{2}$-BPS) critical points.* By evaluating the Hodge-decomposition identity (3.4.1.30) along the constraints (3.1.1.1.1), one obtains

$$\begin{aligned} &\mathcal{H}_3 = -i\left[e^K\left(W\overline{\Omega}_3 - \overline{W}\Omega_3\right)\right]_{\frac{1}{2}-BPS} = \\ &= -2Im\left(\overline{Z}\hat{\Omega}_3\right)_{\frac{1}{2}-BPS}. \end{aligned} \tag{3.4.2.1}$$

Eq. (3.4.2.1) is the translation, for Type IIB on CY_3, of Eqs. (3.3.2.1) and (3.3.2.2), which in turn are equivalent, purely algebraic forms of the $\frac{1}{2}$-BPS extremal BH AEs, given by the (partly differential) conditions (3.1.1.1.1). By recalling Eqs. (3.4.1.14) and (3.4.1.15), Eq. (3.4.2.1) implies that at $\frac{1}{2}$-BPS critical points of V_{BH} the real flux 3-form $\mathcal{H}_3$ of Type IIB on CY_3 has vanishing components along the Dalbeault third cohomologies $H^{2,1}(CY_3)$ and $H^{1,2}(CY_3)$. This can be understood easily by recalling Eq. (3.4.1.23):

$$D_iW = 0,\ \forall i \Longleftrightarrow \left\{\begin{array}{c} \int_{CY_3}\mathcal{H}_3\wedge D_i\Omega_3 = 0,\ \forall i; \\ \Updownarrow \\ \int_{CY_3}\mathcal{H}_3\wedge \overline{D}_{\overline{i}}\overline{\Omega}_3 = 0,\ \forall \overline{i}. \end{array}\right. \tag{3.4.2.2}$$

Thus, at $\frac{1}{2}$-BPS critical points of V_{BH} $\mathcal{H}_3$ is *"orthogonal"* (in the sense of Eq. (3.4.2.2), as understood below, as well) to *all* the 3-forms which are basis elements of $H^{2,1}(CY_3)$ and $H^{1,2}(CY_3)$.

Consequently, the complete supersymmetry breaking at the horizon of (static, spherically symmetric, asymptotically flat) extremal BHs in $\mathcal{N}=2$, $d=4$ supergravity as low-energy, effective theory of Type IIB on CY_3 can be traced back to the non-vanishing *"intersections"* (defined by Eqs. (3.4.1.23) and (3.4.1.24)) of $\mathcal{H}_3$ with $H^{2,1}(CY_3)$ and $H^{1,2}(CY_3)$. Moreover, in light of Eq. (3.4.1.22), the $\frac{1}{2}$-BPS non-degeneracy condition $W_{\frac{1}{2}-BPS}\neq 0$ corresponds to a condition of non(-complete)-*"orthogonality"* between $\mathcal{H}_3$ and Ω_3, basis of $H^{3,0}(CY_3)$:

$$W \neq 0 \Longleftrightarrow \left\{ \begin{array}{c} \int_{CY_3} \mathcal{H}_3 \wedge \Omega_3 \neq 0; \\ \Updownarrow \\ \int_{CY_3} \mathcal{H}_3 \wedge \overline{\Omega}_3 \neq 0. \end{array} \right. \tag{3.4.2.3}$$

II) *Non-BPS $Z \neq 0$ critical points.* By evaluating the Hodge-decomposition identity (3.4.1.30) along the constraints (3.1.1.2.1) and (3.1.1.2.2), one obtains

$$\mathcal{H}_3 = 2Im\left[Z\overline{\hat{\Omega}}_3 + \frac{i}{2}\frac{Z}{|Z|^2}C_{ikl}g^{i\bar{j}}g^{k\bar{k}}g^{l\bar{l}}\left(\overline{D}_{\bar{k}}\overline{Z}\right)\left(\overline{D}_{\bar{l}}\overline{Z}\right)\overline{D}_{\bar{j}}\overline{\hat{\Omega}}_3\right]_{non-BPS,Z\neq 0}, \tag{3.4.2.4}$$

Eq. (3.4.2.4) is the translation, for Type IIB on CY_3, of Eqs. (3.3.2.2) and (3.3.2.3), which in turn are equivalent forms of the non-BPS $Z \neq 0$ extremal BH AEs, given by the (partly differential) conditions (3.1.1.2.1) and (3.1.1.2.2).

By recalling Eqs. (3.4.1.14) and (3.4.1.15), Eq. (3.4.2.4) implies that at the non-BPS $Z \neq 0$ critical points of V_{BH} the real flux 3-form $\mathcal{H}_3$ of type IIB on CY_3 has components along $H^{0,3}(CY_3)$ and $H^{2,1}(CY_3)$ with the same holomorphicity in the holomorphic central charge Z. In other words, such components can be expressed only in terms of Z and D_iZ, without using $\overline{Z}$ and $\overline{D}_{\bar{i}}\overline{Z}$. Such a fact does not happen in a generic point of the CS moduli space of CY_3, as it is seen from the global Hodge-decomposition identity (3.4.1.30). As it is evident, the price to be paid in order to obtain the same holomorphicity in Z at the non-BPS $Z \neq 0$ critical points of V_{BH} is the fact that the component of $\mathcal{H}_3$ along $H^{2,1}(CY_3)$ is not linear in some covariant derivative of Z any more, also explicitly depending on the rank-3 covariantly antiholomorphic tensor $\overline{C}_{\overline{ijk}}$. By recalling Eqs. (3.4.1.22)-(3.4.1.25), this can be understood by considering the translation of Eq. (3.1.1.2.2) in the language of (3-)forms of Type IIB on CY_3:

$$\int_{CY_3}\mathcal{H}_3\wedge\overline{D}_{\bar{i}}\overline{\Omega}_3 = \frac{i}{2\int_{CY_3}\mathcal{H}_3\wedge\Omega_3}\overline{C}_{\overline{ijk}}g^{l\bar{j}}g^{m\bar{k}}\left(\int_{CY_3}\mathcal{H}_3\wedge D_l\Omega_3\right)\int_{CY_3}\mathcal{H}_3\wedge D_m\Omega_3, \quad \forall \bar{i}=\bar{1},...,\overline{n_V}. \tag{3.4.2.5}$$

Eq. (3.4.2.5), holding at non-BPS $Z \neq 0$ critical points of V_{BH}, expresses the *"intersections"* of $\mathcal{H}_3$ with $H^{1,2}(CY_3)$ (*i.e.* the components of $\mathcal{H}_3$ along $H^{2,1}(CY_3)$; see Eq. (3.4.1.28)) non-linearly in terms of *"intersections"* of $\mathcal{H}_3$ with $H^{3,0}(CY_3)$ and $H^{2,1}(CY_3)$, which can be all expressed only in terms of Z and D_iZ, without using $\overline{Z}$ and $\overline{D}_{\bar{i}}\overline{Z}$.

III) *Non-BPS $Z=0$ critical points.* By evaluating the Hodge-decomposition identity (3.4.1.30) along the constraints (3.1.1.3.1) and (3.1.1.3.2), one obtains

$$\begin{aligned}\mathcal{H}_3 &= -ie^K\left[-g^{i\bar{j}}\left(\partial_i W\right)\overline{D}_{\bar{j}}\overline{\Omega}_3 + g^{j\bar{i}}\left(\overline{\partial}_{\bar{i}}\overline{W}\right)D_j\Omega_3\right] = \\ &= 2Im\left[g^{j\bar{i}}\left(\overline{\partial}_{\bar{i}}\overline{Z}\right)D_j\hat{\Omega}_3\right],\end{aligned} \tag{3.4.2.6}$$

Eq. (3.4.2.6) is the translation, for Type IIB on CY_3, of Eqs. (3.3.2.4) and (3.3.2.5), which in turn are equivalent forms of the non-BPS $Z=0$ extremal BH AEs, given by the (partly differential) conditions (3.1.1.3.1) and (3.1.1.3.2). By recalling Eqs. (3.4.1.14) and (3.4.1.15), Eq. (3.4.2.6) implies that at non-BPS $Z=0$ critical points of V_{BH}, in an opposite fashion with respect to the case of $\frac{1}{2}$-BPS critical points of V_{BH}, the real flux 3-form $\mathcal{H}_3$ of Type IIB on CY_3 has vanishing components along the Dalbeault third cohomologies $H^{3,0}(CY_3)$ and $H^{0,3}(CY_3)$. This can be understood easily by recalling Eq. (3.4.1.22):

$$W = 0 \Longleftrightarrow \left\{\begin{array}{c}\int_{CY_3}\mathcal{H}_3\wedge\Omega_3 = 0; \\ \Updownarrow \\ \int_{CY_3}\mathcal{H}_3\wedge\overline{\Omega}_3 = 0.\end{array}\right. \tag{3.4.2.7}$$

Thus, at non-BPS $Z=0$ critical points of V_{BH} $\mathcal{H}_3$ is *"orthogonal"* (in the sense of Eq. (3.4.2.7), as understood below, as well) to Ω_3 and $\overline{\Omega}_3$, basis of $H^{3,0}(CY_3)$ and $H^{0,3}(CY_3)$, respectively. Moreover, in light of Eqs. (3.4.1.23) and (3.4.1.24), the non-BPS $Z=0$ non-degeneracy condition (*at least* for strictly positive-definite $g_{i\bar{j}}$ at the considered critical points of V_{BH})

$$\left(D_i W\right)_{non-BPS,Z=0} \neq 0, \textit{ at least} \text{ for some } i \in \{1,...,n_V\} \tag{3.4.2.8}$$

corresponds to a condition of non(-complete)-*"orthogonality"* between $\mathcal{H}_3$ and the $D_i\Omega_3$s, basis elements of $H^{2,1}(CY_3)$:

$$D_i W \neq 0, \textit{ at least} \text{ for some } i \in \{1,...,n_V\} \Longleftrightarrow \left\{\begin{array}{c}\int_{CY_3}\mathcal{H}_3\wedge D_i\Omega_3 \neq 0,, \textit{ at least} \text{ for some } i \in \{1,...,n_V\}; \\ \Updownarrow \\ \int_{CY_3}\mathcal{H}_3\wedge\overline{D}_{\bar{i}}\overline{\Omega}_3 \neq 0, \textit{ at least} \text{ for some } \bar{i} \in \left\{\bar{1},...,\overline{n_V}\right\}.\end{array}\right. \tag{3.4.2.9}$$

4 Flux Vacua Attractor Equations in $\mathcal{N}=1$, $d=4$ Supergravity from Type IIB on $\frac{CY_3\times T^2}{\mathbb{Z}_2}$

4.1 CY_3 Orientifolds

We consider Type IIB superstring theory compactified on a CY_3 orientifold with O3/O7-planes (as in the GPK-KKLT model [109, 80]), determining an $\mathcal{N}=1$, $d=4$ supergravity as effective, low-energy theory. Within such a framework, we will derive FV AEs[14], similarly to what done in Subsect. 3.4 for extremal

[14] In [110] the FV Attractor Mechanism has been shown to act also in the landscape of non-Kähler vacua emerging in the flux compactifications of heterotic superstrings.

BH AEs in effective $\mathcal{N}=2$, $d=4$ supergravity from Type IIB on CY_3. We will mainly follow [9], [84] and [85]. In our treatment, the relevant moduli moduli space M of the CY_3 orientifold is the one composed by the (direct) product of the CS moduli space (of complex dimension $h_{2,1} \equiv dim\left(H^{2,1}\left(CY_3\right)\right)$, which is a SK manifold, and the 1-dim. Kähler manifold parameterized by the universal axion-dilaton. We will denote the CS moduli by $\left(x^i,\overline{x}^{\overline{i}}\right)_{i=1,\ldots,h_{2,1}} \equiv \left(t^i,\overline{t}^{\overline{i}}\right)_{i=1,\ldots,h_{2,1}}$ (not to be confused with the projective coordinates in the SK CS moduli space) and the axion-dilaton by $\tau \equiv t^0$:

$$M=\mathcal{M}_{t^0}\otimes\mathcal{M}_{CS}. \tag{4.1.1}$$

No Kähler structure (KS) moduli will be considered in our treatment of the classical FV Attractor Mechanism; indeed, in the considered framework the stabilization of KS moduli requires quantum perturbative or non-perturbative mechanisms, such as worldsheet instantons and gaugino condensation (see *e.g.* [80]).

4.1.1 Vielbein and Metric Tensor in the Moduli Space

We start by defining the structure of the $(h_{2,1}+1)$-dim. Kähler manifold spanned by the CS moduli and the axion-dilaton. Its Kähler potential can be written as follows ($\Lambda=1,\ldots,h_{2,1},h_{2,1}+1$ throughout[15]):

$$\begin{aligned}
&K\left(t,\overline{t}\right)=-ln\left[-i\left(t^0-\overline{t}^{\overline{0}}\right)\right]-ln\left[i\int_{CY_3}\Omega_3\left(x\right)\wedge\overline{\Omega}_3\left(\overline{x}\right)\right]=\\
&=-ln\left[\int_{CY_3}\left[t^0\Omega_3\left(x\right)\wedge\overline{\Omega}_3\left(\overline{x}\right)-\Omega_3\left(x\right)\wedge\overline{t}^{\overline{0}}\overline{\Omega}_3\left(\overline{x}\right)\right]\right]=\\
&=-ln\left[\left(t^0-\overline{t}^{\overline{0}}\right)\left(\overline{X}^{\Lambda}\left(\overline{x}\right)F_{\Lambda}\left(x\right)-X^{\Lambda}\left(x\right)\overline{F}_{\Lambda}\left(\overline{x}\right)\right)\right],
\end{aligned} \tag{4.1.1.1}$$

where Ω_3 is the holomorphic $(3,0)$-form defined on CY_3. Thus, one can write:

$$\left\{\begin{array}{l}
K\left(t,\overline{t}\right)=K_1\left(t^0,\overline{t}^{\overline{0}}\right)+K_3\left(x,\overline{x}\right);\\
\\
K_1\left(t^0,\overline{t}^{\overline{0}}\right)\equiv -ln\left[-i\left(t^0-\overline{t}^{\overline{0}}\right)\right];\\
\\
K_3\left(x,\overline{x}\right)\equiv -ln\left[i\left(\overline{X}^{\Lambda}\left(\overline{x}\right)F_{\Lambda}\left(x\right)-X^{\Lambda}\left(x\right)\overline{F}_{\Lambda}\left(\overline{x}\right)\right)\right].
\end{array}\right. \tag{4.1.1.2}$$

The reality condition on K_1 and K_3 yields the conditions

$$Imt^0>0,\ Im\left(X^{\Lambda}\left(x\right)\overline{F}_{\Lambda}\left(\overline{x}\right)\right)>0. \tag{4.1.1.3}$$

The metric of the whole moduli space is given by ($a=0,1,\ldots,h_{2,1}$ throughout)

$$g_{a\overline{b}}\left(t,\overline{t}\right)=\overline{\partial}_{\overline{b}}\partial_a K\left(t,\overline{t}\right)=\overline{\partial}_{\overline{b}}\partial_a\left[K_1\left(t^0,\overline{t}^{\overline{0}}\right)+K_3\left(x,\overline{x}\right)\right], \tag{4.1.1.4}$$

[15]Notice the different range of the symplectic (capital Greek) indices in the present treatment of Type IIB on CY_3 orientifold with O3/O7-planes with respect to the range $0,1,\ldots,h_{2,1}$ of the previous treatment of Type IIB on CY_3. In general, the reference to the graviphoton degree of freedom "0" is lost, due to the orientifolding truncation of the low-energy, effective supergravity.

yielding

$$\begin{cases} g_{0\overline{0}} = -\left(\overline{t}^{\overline{0}} - t^0\right)^{-2} = e^{2K_1\left(t^0,\overline{t}^{\overline{0}}\right)}; \\ g_{0\overline{i}} = 0 = g_{i\overline{0}}; \\ g_{i\overline{j}} = \overline{\partial}_{\overline{j}}\partial_i K_3\left(x,\overline{x}\right). \end{cases} \tag{4.1.1.5}$$

In our treatment we will make extensive use of the local "flat" coordinates in M (denoted by capital indices $A = \underline{0}, \underline{1}, ..., \underline{h_{2,1}}$ throughout), defined as usual by ($g_{a\overline{b}}g^{a\overline{c}} = \delta_{\overline{b}}^{\overline{c}}$, $g_{a\overline{b}}g^{c\overline{b}} = \delta_a^c$)

$$g_{a\overline{b}}\left(t,\overline{t}\right) \equiv e_a^A\left(t,\overline{t}\right)\overline{e}_{\overline{b}}^{\overline{B}}\left(t,\overline{t}\right)\delta_{A\overline{B}} \Leftrightarrow g^{a\overline{b}}\left(t,\overline{t}\right) \equiv e_A^a\left(t,\overline{t}\right)\overline{e}_{\overline{B}}^{\overline{b}}\left(t,\overline{t}\right)\delta^{A\overline{B}}, \tag{4.1.1.6}$$

where $e_a^A\left(t,\overline{t}\right)$ is the local vielbein in M, and $e_A^a\left(t,\overline{t}\right)$ is its inverse ($e_a^A e_A^b = \delta_a^b$, $e_a^A e_B^a = \delta_B^A$). Due to Eqs. (4.1.1.5), the $h_{2,1}^2 + 2h_{2,1} + 1$ components of the vielbein $e_a^A = \left\{e_0^{\underline{0}}, e_i^{\underline{0}}, e_0^I, e_i^I\right\}$ ($I = \underline{1}, ..., \underline{h_{2,1}}$ throughout), defined by Eq. (4.1.1.6), satisfy the following set of Eqs.:

$$\begin{cases} \left|e_0^{\underline{0}}\left(t,\overline{t}\right)\right|^2 + e_0^I\left(t,\overline{t}\right)\overline{e}_{\overline{0}}^{\overline{J}}\left(t,\overline{t}\right)\delta_{I\overline{J}} = -\left(\overline{t}^{\overline{0}} - t^0\right)^{-2}; \\ e_0^{\underline{0}}\left(t,\overline{t}\right)\overline{e}_{\overline{i}}^{\overline{\underline{0}}}\left(t,\overline{t}\right) + e_0^I\left(t,\overline{t}\right)\overline{e}_{\overline{i}}^{\overline{J}}\left(t,\overline{t}\right)\delta_{I\overline{J}} = 0; \\ e_i^{\underline{0}}\left(t,\overline{t}\right)\overline{e}_{\overline{j}}^{\overline{\underline{0}}}\left(t,\overline{t}\right) + e_i^I\left(t,\overline{t}\right)\overline{e}_{\overline{j}}^{\overline{J}}\left(t,\overline{t}\right)\delta_{I\overline{J}} = \overline{\partial}_{\overline{j}}\partial_i K_3\left(x,\overline{x}\right), \end{cases} \tag{4.1.1.7}$$

admitting as a solution[16]:

$$\begin{cases} \left|e_0^{\underline{0}}\left(t,\overline{t}\right)\right|^2 = -\left(\overline{t}^{\overline{0}} - t^0\right)^{-2} \Leftarrow e_0^{\underline{0}}\left(t,\overline{t}\right) = \left(\overline{t}^{\overline{0}} - t^0\right)^{-1} = ie^{K_1\left(t^0,\overline{t}^{\overline{0}}\right)} = e_0^{\underline{0}}\left(t^0,\overline{t}^{\overline{0}}\right); \\ e_0^I\left(t,\overline{t}\right) = 0, \ \forall I = \underline{1}, ..., \underline{h_{2,1}}; \\ e_i^{\underline{0}}\left(t,\overline{t}\right) = 0, \ \forall i = 1, ..., h_{2,1}; \\ e_i^I\left(t,\overline{t}\right)\overline{e}_{\overline{j}}^{\overline{J}}\left(t,\overline{t}\right)\delta_{I\overline{J}} = \overline{\partial}_{\overline{j}}\partial_i K_3\left(x,\overline{x}\right). \end{cases} \tag{4.1.1.8}$$

By inverting Eq. (4.1.1.6) one gets

$$\delta_{A\overline{B}} = e_A^a\left(t,\overline{t}\right)\overline{e}_{\overline{B}}^{\overline{b}}\left(t,\overline{t}\right)g_{a\overline{b}}\left(t,\overline{t}\right) \Leftrightarrow \delta^{A\overline{B}} = e_a^A\left(t,\overline{t}\right)\overline{e}_{\overline{b}}^{\overline{B}}\left(t,\overline{t}\right)g^{a\overline{b}}\left(t,\overline{t}\right), \tag{4.1.1.9}$$

which by Eq. (4.1.1.5) implies that the $h_{2,1}^2 + 2h_{2,1} + 1$ components of the inverse vielbein $e_A^a =$

[16] Notice that the solutions given by Eqs. (4.1.1.8) and (4.1.1.11) are clearly not unique. Indeed, for a given metric, one can always transform the vielbein and its inverse by a Lorentz transformation, which however will not affect the metric itself.

$\left\{ e_{\underline{0}}^{0}, e_{\underline{0}}^{i}, e_{I}^{0}, e_{I}^{i} \right\}$, defined by Eq. (4.1.1.6), satisfy the set following set of Eqs.:

$$\left\{ \begin{array}{l} -\left(\overline{t}^{\overline{0}}-t^{0}\right)^{-2}\left|e_{\underline{0}}^{0}\left(t,\overline{t}\right)\right|^{2}+e_{\underline{0}}^{i}\left(t,\overline{t}\right)\overline{e}_{\underline{\overline{0}}}^{\overline{j}}\left(t,\overline{t}\right)\overline{\partial}_{\overline{j}}\partial_{i}K_{3}\left(x,\overline{x}\right)=1; \\ \\ -\left(\overline{t}^{\overline{0}}-t^{0}\right)^{-1}\overline{e}_{\overline{I}}^{\overline{0}}\left(t,\overline{t}\right)+e_{\underline{0}}^{i}\left(t,\overline{t}\right)\overline{e}_{\overline{I}}^{\overline{j}}\left(t,\overline{t}\right)\overline{\partial}_{\overline{j}}\partial_{i}K_{3}\left(x,\overline{x}\right)=0; \\ \\ -\left(\overline{t}^{\overline{0}}-t^{0}\right)^{-2}e_{I}^{0}\left(t,\overline{t}\right)\overline{e}_{\overline{J}}^{\overline{0}}\left(t,\overline{t}\right)+e_{I}^{i}\left(t,\overline{t}\right)\overline{e}_{\overline{J}}^{\overline{j}}\left(t,\overline{t}\right)\overline{\partial}_{\overline{j}}\partial_{i}K_{3}\left(x,\overline{x}\right)=\delta_{I\overline{J}}, \end{array} \right. \tag{4.1.1.10}$$

admitting as a solution:

$$\left\{ \begin{array}{l} \left|e_{\underline{0}}^{0}\left(t,\overline{t}\right)\right|^{2}=-\left(\overline{t}^{\overline{0}}-t^{0}\right)^{2}=\left|e_{0}^{\underline{0}}\left(t,\overline{t}\right)\right|^{-2}; \\ \Uparrow \\ e_{\underline{0}}^{0}\left(t,\overline{t}\right)=\left(\overline{t}^{\overline{0}}-t^{0}\right)=-ie^{-K_{1}\left(t^{0},\overline{t}^{\overline{0}}\right)}=\left[e_{0}^{\underline{0}}\left(t^{0},\overline{t}^{\overline{0}}\right)\right]^{-1}=e_{\underline{0}}^{0}\left(t^{0},\overline{t}^{\overline{0}}\right); \\ \\ e_{\underline{0}}^{i}\left(t,\overline{t}\right)=0,\ \forall i=1,...,h_{2,1}; \\ \\ e_{I}^{0}\left(t,\overline{t}\right)=0,\ \forall I=\underline{1},...,\underline{h_{2,1}}; \\ \\ e_{I}^{i}\left(t,\overline{t}\right)\overline{e}_{\overline{J}}^{\overline{j}}\left(t,\overline{t}\right)\overline{\partial}_{\overline{j}}\partial_{i}K_{3}\left(x,\overline{x}\right)=\delta_{I\overline{J}}, \end{array} \right. \tag{4.1.1.11}$$

implying, by Eq. (4.1.1.6), that the components of the inverse metric tensor of M read as follows:

$$\left\{ \begin{array}{l} g^{0\overline{0}}=-\left(\overline{t}^{\overline{0}}-t^{0}\right)^{2}=e^{-2K_{1}\left(t^{0},\overline{t}^{\overline{0}}\right)}=\left(g_{0\overline{0}}\right)^{-1}; \\ \\ g^{0\overline{i}}=0=g^{i\overline{0}}; \\ \\ g^{i\overline{j}}:g^{i\overline{j}}\overline{\partial}_{\overline{j}}\partial_{k}K_{3}\left(x,\overline{x}\right)=\delta_{k}^{i},\ g^{i\overline{j}}\overline{\partial}_{\overline{k}}\partial_{i}K_{3}\left(x,\overline{x}\right)=\delta_{\overline{k}}^{\overline{j}}. \end{array} \right. \tag{4.1.1.12}$$

Moreover, it should be noticed that actually $e_{i}^{I}=e_{i}^{I}\left(x,\overline{x}\right)$ and $e_{I}^{i}=e_{I}^{i}\left(x,\overline{x}\right)$, as obtained by differentiating with respect to the axion-dilaton t^{0} the fourth Eq. of the systems of solutions (4.1.1.8) and

(4.1.1.11), respectively:

$$
\begin{gathered}
\left\{\left[\partial_0 e_i^I\left(t,\overline{t}\right)\right]\overline{e}_{\overline{j}}^{\overline{J}}\left(t,\overline{t}\right)+e_i^I\left(t,\overline{t}\right)\partial_0\overline{e}_{\overline{j}}^{\overline{J}}\left(t,\overline{t}\right)\right\}\delta_{I\overline{J}}=0;\\
\Updownarrow\\
\left\{\begin{array}{l}
\partial_0 e_i^I\left(t,\overline{t}\right)=0,\\
\\
\partial_0\overline{e}_{\overline{i}}^{\overline{I}}\left(t,\overline{t}\right)=0\Leftrightarrow\overline{\partial}_{\overline{0}}e_i^I\left(t,\overline{t}\right)=0;
\end{array}\right.\\
\Updownarrow\\
e_i^I=e_i^I\left(x,\overline{x}\right);
\end{gathered}
$$

$$
\begin{gathered}
\left\{\left[\partial_0 e_I^i\left(t,\overline{t}\right)\right]\overline{e}_{\overline{J}}^{\overline{j}}\left(t,\overline{t}\right)+e_I^i\left(t,\overline{t}\right)\partial_0\overline{e}_{\overline{J}}^{\overline{j}}\left(t,\overline{t}\right)\right\}\overline{\partial}_{\overline{j}}\partial_i K_3\left(x,\overline{x}\right)=0;\\
\Updownarrow\\
\left\{\begin{array}{l}
\partial_0 e_I^i\left(t,\overline{t}\right)=0,\\
\\
\partial_0\overline{e}_{\overline{I}}^{\overline{i}}\left(t,\overline{t}\right)\Leftrightarrow\overline{\partial}_{\overline{0}}e_I^i\left(t,\overline{t}\right)=0;
\end{array}\right.\\
\Updownarrow\\
e_I^i=e_I^i\left(x,\overline{x}\right).
\end{gathered}
\tag{4.1.1.13}
$$

In the following treatment, we will use the solutions (4.1.1.8) and (4.1.1.11) of the systems of Eqs. (4.1.1.7) and (4.1.1.10), respectively, *i.e.* we will assume that a system of local "flat" coordinates in M defined by Eqs. (4.1.1.6) and (4.1.1.9) always exists such that the corresponding vielbein and its inverse are given by Eqs. (4.1.1.8) and (4.1.1.11) (implemented by Eqs. (4.1.1.13)), in turn consistent with the covariant and contravariant metric tensor of M given by Eqs. (4.1.1.5) and (4.1.1.12), respectively.

4.1.2 1-, 3- and 4-Forms on $\frac{CY_3\times T^2}{\mathbb{Z}_2}$

Next, we introduce the Ramond-Ramond (RR) and Neveu-Schwarz-Neveu-Schwarz (NSNS) flux 3-forms of Type IIB on CY_3 orientifold (with O3/O7-planes) as follows:

$$
\begin{gathered}
RR:\mathfrak{F}_3\equiv p_f^\Lambda\alpha_\Lambda-q_{f|\Lambda}\beta^\Lambda\in H^3\left(CY_3,\mathbb{R}\right);\\
NSNS:\mathfrak{H}_3\equiv p_h^\Lambda\alpha_\Lambda-q_{h|\Lambda}\beta^\Lambda\in H^3\left(CY_3,\mathbb{R}\right),
\end{gathered}
\tag{4.1.2.1}
$$

where we introduced the $1\times(2h_{2,1}+2)$ symplectic vector of RR and NSNS fluxes (charges), respectively:

$$
\begin{gathered}
Q_{RR}\equiv\left(p_f^\Lambda,q_{f|\Lambda}\right);\\
Q_{NSNS}\equiv\left(p_h^\Lambda,q_{h|\Lambda}\right),
\end{gathered}
\tag{4.1.2.2}
$$

and $\left\{\alpha_\Lambda,\beta^\Lambda\right\}$ is the b_3-dim. real (manifestly symplectic-covariant) basis of the third real cohomology $H^3\left(CY_3,\mathbb{R}\right)$, satisfying Eq. (3.4.1.1). In the considered framework, the flux 3-forms defined by Eq. (4.1.2.1) can be unified in the t^0-dependent, complex flux 3-form

$$
\mathfrak{G}_3\left(t^0\right)\equiv\mathfrak{F}_3-t^0\mathfrak{H}_3=\left(p_f^\Lambda-t^0p_h^\Lambda\right)\alpha_\Lambda-\left(q_{f|\Lambda}-t^0q_{h|\Lambda}\right)\beta^\Lambda\in H^3\left(CY_3,\mathbb{C};t^0\right),
\tag{4.1.2.3}
$$

thus determining the GVW $\mathcal{N}=1$, $d=4$ holomorphic superpotential as follows:

$$
\begin{aligned}
W\left(t\right)&\equiv\int_{CY_3}\mathfrak{G}_3\left(t^0\right)\wedge\Omega_3\left(x\right)=\int_{CY_3}\mathfrak{F}_3\wedge\Omega_3\left(x\right)-t^0\int_{CY_3}\mathfrak{H}_3\wedge\Omega_3\left(x\right)=\\
&=q_{f|\Lambda}X^\Lambda\left(x\right)-p_f^\Lambda F_\Lambda\left(x\right)+q_{h|\Lambda}\left(-t^0X^\Lambda\left(x\right)\right)-p_h^\Lambda\left(-t^0F_\Lambda\left(x\right)\right).
\end{aligned}
\tag{4.1.2.4}
$$

The second line of Eq. (4.1.2.4) suggests to redefine the holomorphic (3,0)-form in the *"NSNS sector"* as follows:

$$\Omega_{3,NS}(t) \equiv -t^0 \Omega_{3,RR}(x) = -t^0 \Omega_3(x). \tag{4.1.2.5}$$

Since in Type IIB on the considered CY_3 orientifold the flux 3-forms $\mathfrak{F}_3$ and $\mathfrak{H}_3$ form the $SL\left(2, H^3(CY_3, \mathbb{R})\right)$-doublet

$$\widehat{F} \equiv \left(\begin{array}{c} \mathfrak{F}_3 \\ \\ \mathfrak{H}_3 \end{array} \right) \in SL\left(2, H^3(CY_3, \mathbb{R})\right), \tag{4.1.2.6}$$

correspondingly, one can introduce the $SL\left(2, H^{3,0}(CY_3; t)\right)$-doublet

$$\Xi(t) \equiv \left(\begin{array}{c} \Xi_1(x) \equiv \Omega_3(x) \\ \\ \Xi_2(t) \equiv -t^0 \Omega_3(x) \end{array} \right) \in SL\left(2, H^{3,0}(CY_3; t)\right). \tag{4.1.2.7}$$

By exploiting such a manifest $SL(2)$-covariance, Eqs. (4.1.1.1) and (4.1.2.4) can be rewritten as follows:

$$K\left(t, \overline{t}\right) = -ln\left[\int_{CY_3}\left[\Xi_1(x) \wedge \overline{\Xi}_2\left(\overline{t}\right) - \Xi_2(t) \wedge \overline{\Xi}_1(\overline{x})\right]\right]; \tag{4.1.2.8}$$

$$W(t) = \int_{CY_3}\left[\mathfrak{F}_3 \wedge \Xi_1(x) + \mathfrak{H}_3 \wedge \Xi_2(t)\right] = \int_{CY_3} \widehat{F}^T \wedge \Xi(t). \tag{4.1.2.9}$$

Thus, the $\mathcal{N}=1$, $d=4$ covariantly holomorphic central charge function of Type IIB on CY_3 orientifold with O3/O7-planes can be introduced:

$$\begin{aligned} Z\left(t, \overline{t}\right) &\equiv e^{\frac{1}{2}K\left(t, \overline{t}\right)} W(t) = e^{\frac{1}{2}K\left(t, \overline{t}\right)} \int_{CY_3} \mathfrak{G}_3\left(t^0\right) \wedge \Omega_3(x) = & (4.1.2.10) \\ &= \frac{\int_{CY_3} \widehat{F}^T \wedge \Xi(t)}{\sqrt{\int_{CY_3}\left[\Xi_1(x) \wedge \overline{\Xi}_2\left(\overline{t}\right) - \Xi_2(t) \wedge \overline{\Xi}_1(\overline{x})\right]}} = & (4.1.2.11) \\ &= \frac{\left(q_{f|\Lambda} - t^0 q_{h|\Lambda}\right) X^\Lambda(x) - \left(p_f^\Lambda - t^0 p_h^\Lambda\right) F_\Lambda(x)}{\sqrt{\left(t^0 - \overline{t}^0\right)\left(\overline{X}^\Lambda(\overline{x}) F_\Lambda(x) - X^\Lambda(x) \overline{F}_\Lambda(\overline{x})\right)}}, & (4.1.2.12) \end{aligned}$$

with Kähler weights $(1,-1)$ with respect to $K\left(t, \overline{t}\right)$:

$$\begin{aligned} D_a Z\left(t, \overline{t}\right) &= \partial_a Z\left(t, \overline{t}\right) + \tfrac{1}{2}\left(\partial_a K\left(t, \overline{t}\right)\right) Z\left(t, \overline{t}\right); \\ \overline{D}_{\overline{a}} Z\left(t, \overline{t}\right) &= \overline{\partial}_{\overline{a}} Z\left(t, \overline{t}\right) - \tfrac{1}{2}\left(\overline{\partial}_{\overline{a}} K\left(t, \overline{t}\right)\right) Z\left(t, \overline{t}\right) = 0. \end{aligned} \tag{4.1.2.13}$$

Now, we can perform an unifying simplification of notation, by using the language of 4-forms on Calabi-Yau 4-folds (CY_4); in such a framework, Type IIB on CY_3 orientifold with O3/O7-planes can be described by 4-forms defined on $CY_4 = \frac{CY_3 \times T^2}{\mathbb{Z}_2}$, where T^2 denotes the "auxiliary" 2-torus, whose complex modulus is the universal-axion dilaton $\tau \equiv t^0$. Thus, beside the b_3-dim. real (manifestly symplectic-covariant) basis $\left\{\alpha_\Lambda, \beta^\Lambda\right\}$ of $H^3(CY_3, \mathbb{R})$ (satisfying Eq. (3.4.1.1)), one can introduce the 2-dim. basis $\{\alpha, \beta\}$ of $H^1\left(T^2, \mathbb{R}\right)$, satisfying

$$\int_{T^2} \alpha \wedge \alpha = 0 = \int_{T^2} \beta \wedge \beta, \quad \int_{T^2} \alpha \wedge \beta = 1, \tag{4.1.2.14}$$

and the holomorphic $(1,0)$-form $\Omega_1\left(t^0\right)$ on T^2:

$$\Omega_1\left(t^0\right) \equiv -t^0\alpha+\beta \in H^{1,0}\left(T^2\right). \tag{4.1.2.15}$$

By recalling Eq. (3.4.1.3), it is thus possible to define an holomorphic $(4,0)$-form on $CY_4(=\frac{CY_3\times T^2}{\mathbb{Z}_2}$, as always understood in treatment below) as follows:

$$\Omega_4(t) \quad\equiv\quad \Omega_1\left(t^0\right)\wedge\Omega_3(x)=X^\Lambda(x)\,\beta\wedge\alpha_\Lambda-t^0X^\Lambda(x)\,\alpha\wedge\alpha_\Lambda-F_\Lambda(x)\,\beta\wedge\beta^\Lambda+t^0F_\Lambda(x)\,\alpha\wedge\beta^\Lambda. \tag{4.1.2.16}$$

Instead of using the complex, t^0-dependent flux 3-form $\mathfrak{G}_3\left(t^0\right)\in H^3\left(CY_3,\mathbb{C};t^0\right)$ defined by Eq. (4.1.2.3), the RR and NSNS flux 3-forms can be unified elegantly by introducing the real flux 4-form

$$\mathfrak{F}_4 \quad\equiv\quad -\alpha\wedge\mathfrak{F}_3+\beta\wedge\mathfrak{H}_3=\left(p_h^\Lambda\beta-p_f^\Lambda\alpha\right)\wedge\alpha_\Lambda-\left(q_{h|\Lambda}\beta-q_{f|\Lambda}\alpha\right)\wedge\beta^\Lambda\in H^4\left(CY_4,\mathbb{R}\right). \tag{4.1.2.17}$$

By using Eqs. (3.4.1.1), (3.4.1.3), (4.1.2.14), (4.1.2.15), (4.1.2.16) and (4.1.2.17), Eqs. (4.1.2.8)-(4.1.2.11) can be elegantly rewritten as follows:

$$K\left(t,\overline{t}\right)=-ln\left(\int_{CY_4}\Omega_4(t)\wedge\overline{\Omega}_4\left(\overline{t}\right)\right); \tag{4.1.2.18}$$

$$W(t)=\int_{CY_4}\mathfrak{F}_4\wedge\Omega_4(t); \tag{4.1.2.19}$$

$$Z\left(t,\overline{t}\right)=e^{\frac{1}{2}K\left(t,\overline{t}\right)}\int_{CY_4}\mathfrak{F}_4\wedge\Omega_4(t)=\frac{\int_{CY_4}\mathfrak{F}_4\wedge\Omega_4(t)}{\sqrt{\int_{CY_4}\Omega_4(t)\wedge\overline{\Omega}_4\left(\overline{t}\right)}}=\int_{CY_4}\mathfrak{F}_4\wedge\hat{\Omega}_4\left(t,\overline{t}\right), \tag{4.1.2.20}$$

where in Eq. (4.1.2.20) we defined the covariantly holomorphic 4-form on CY_4:

$$\begin{aligned}&\hat{\Omega}_4\left(t,\overline{t}\right)\equiv e^{\frac{1}{2}K\left(t,\overline{t}\right)}\Omega_4(t)=e^{\frac{1}{2}K_1\left(t^0,\overline{t}^{\overline{0}}\right)}e^{\frac{1}{2}K_3(x,\overline{x})}\Omega_1\left(t^0\right)\wedge\Omega_3(x)=\hat{\Omega}_1\left(t^0,\overline{t}^{\overline{0}}\right)\wedge\hat{\Omega}_3\left(x,\overline{x}\right);\\&\hat{\Omega}_1\left(t^0,\overline{t}^{\overline{0}}\right)\equiv e^{\frac{1}{2}K_1\left(t^0,\overline{t}^{\overline{0}}\right)}\Omega_1\left(t^0\right);\end{aligned} \tag{4.1.2.21}$$

$\hat{\Omega}_3\left(x,\overline{x}\right)$ is the covariantly holomorphic 3-form on CY_3, defined by Eq. (3.4.1.32); it has Kähler weights $(1,-1)$ with respect to the Kähler potential $K_3\left(x,\overline{x}\right)$ of the SK CS moduli space $\mathcal{M}_{CS}$ of CY_3:

$$\begin{aligned}&D_i\hat{\Omega}_3=\left\{\begin{array}{l}\partial_i\hat{\Omega}_3+\frac{1}{2}\left(\partial_iK_3\right)\hat{\Omega}_3=e^{\frac{1}{2}K_3}D_i\Omega_3=\\[2ex]=\frac{1}{\sqrt{i\left(\overline{X}^\Delta F_\Delta-X^\Delta\overline{F}_\Delta\right)}}\left\{\begin{array}{l}\left[\partial_iX^\Lambda-\frac{\left(\overline{X}^\Sigma\partial_iF_\Sigma-\left(\partial_iX^\Sigma\right)\overline{F}_\Sigma\right)}{\overline{X}^\Xi F_\Xi-X^\Xi\overline{F}_\Xi}X^\Lambda\right]\alpha_\Lambda+\\[2ex]-\left[\partial_iF_\Lambda-\frac{\left(\overline{X}^\Sigma\partial_iF_\Sigma-\left(\partial_iX^\Sigma\right)\overline{F}_\Sigma\right)}{\overline{X}^\Xi F_\Xi-X^\Xi\overline{F}_\Xi}F_\Lambda\right]\beta^\Lambda\end{array}\right\}\end{array}\right.;\\&\overline{D}_{\overline{i}}\hat{\Omega}_3=\overline{\partial}_{\overline{i}}\hat{\Omega}_3-\frac{1}{2}\left(\overline{\partial}_{\overline{i}}K_3\right)\hat{\Omega}_3=0;\\&D_iD_j\hat{\Omega}_3=iC_{ijk}g^{k\overline{l}}\overline{D}_{\overline{l}}\overline{\hat{\Omega}}_3;\quad\overline{D}_{\overline{i}}D_j\hat{\Omega}_3=g_{j\overline{i}}\hat{\Omega}_3;\\&D_0\hat{\Omega}_3=0;\quad\overline{D}_{\overline{0}}\hat{\Omega}_3=0.\end{aligned} \tag{4.1.2.22}$$

On the other hand, $\hat{\Omega}_1\left(t^0,\overline{t}^{\overline{0}}\right)$ is the covariantly holomorphic 1-form on T^2, defined by the second line of Eq. (4.1.2.21); it has Kähler weights $(1,-1)$ with respect to the Kähler potential $K_1\left(t^0,\overline{t}^{\overline{0}}\right)$ of the Kähler 1-dim. moduli space $\mathcal{M}_{t^0}$ of T^2, spanned by the universal axion-dilaton $\tau\equiv t^0$:

$$\begin{gathered}D_0\hat{\Omega}_1=\partial_0\hat{\Omega}_1+\tfrac{1}{2}\left(\partial_0K_1\right)\hat{\Omega}_1=e^{\frac{1}{2}K_1}D_0\Omega_1=ie^{K_1}\overline{\hat{\Omega}}_1=\left(\overline{t}^{\overline{0}}-t^0\right)^{-1}\overline{\hat{\Omega}}_1;\\ \Updownarrow\\ \overline{\hat{\Omega}}_1=-ie^{-K_1}D_0\hat{\Omega}_1\Leftrightarrow\hat{\Omega}_1=ie^{-K_1}\overline{D}_{\overline{0}}\overline{\hat{\Omega}}_1\Leftrightarrow\overline{D}_{\overline{0}}\overline{\hat{\Omega}}_1=-ie^{K_1}\hat{\Omega}_1;\end{gathered}$$

$$\begin{aligned}&\overline{D}_{\overline{0}}\hat{\Omega}_1=\overline{\partial}_{\overline{0}}\hat{\Omega}_1-\tfrac{1}{2}\left(\overline{\partial}_{\overline{0}}K_1\right)\hat{\Omega}_1=0;\\&D_0D_0\hat{\Omega}_1=0;\ \ \overline{D}_{\overline{0}}D_0\hat{\Omega}_1=g_{0\overline{0}}\hat{\Omega}_1=e^{2K_1}\hat{\Omega}_1=-\left(\overline{t}^{\overline{0}}-t^0\right)^{-2}\hat{\Omega}_1;\\&D_i\hat{\Omega}_1=0;\ \ \overline{D}_{\overline{i}}\hat{\Omega}_1=0.\end{aligned}\tag{4.1.2.23}$$

Resultingly, the covariantly holomorphic 4-form $\hat{\Omega}_4\left(t,\overline{t}\right)$ on $CY_4=\frac{CY_3\times T^2}{\mathbb{Z}_2}$, defined by the first line of Eq. (4.1.2.21), has Kähler weights $(1,-1)$ with respect to the whole Kähler potential $K\left(t,\overline{t}\right)=K_1\left(t^0,\overline{t}^{\overline{0}}\right)+K_3\left(x,\overline{x}\right)$ of the $(h_{2,1}+1)$-dim. moduli space $M=\mathcal{M}_{t^0}\otimes\mathcal{M}_{CS}$. of CY_4 (recall Eq. (4.1.1)):

$$\begin{aligned}&D_a\hat{\Omega}_4\left(t,\overline{t}\right)=\partial_a\hat{\Omega}_4\left(t,\overline{t}\right)+\tfrac{1}{2}\left(\partial_aK\left(t,\overline{t}\right)\right)\hat{\Omega}_4\left(t,\overline{t}\right);\\&\overline{D}_{\overline{a}}\hat{\Omega}_4\left(t,\overline{t}\right)=\overline{\partial}_{\overline{a}}\hat{\Omega}_4\left(t,\overline{t}\right)-\tfrac{1}{2}\left(\overline{\partial}_{\overline{a}}K\left(t,\overline{t}\right)\right)\hat{\Omega}_4\left(t,\overline{t}\right)=0,\end{aligned}\tag{4.1.2.24}$$

implying that

$$D_b\overline{D}_{\overline{a}}\hat{\Omega}_4\left(t,\overline{t}\right)=g_{b\overline{a}}\hat{\Omega}_4\left(t,\overline{t}\right).\tag{4.1.2.25}$$

4.1.3 Hodge Decomposition of $\mathfrak{F}_4$

Now, in order to derive the Hodge-decomposition[17] of the real flux 4-form $\mathfrak{F}_4$, we have to determine all the possible independent 4-forms on $CY_4(=\frac{CY_3\times T^2}{\mathbb{Z}_2}$, as always understood throughout). Due to Eqs. (4.1.2.24) and (4.1.2.25), it is easy to realize that, up to the third order of covariant differentiation included, the possible independent 4-forms ($(1,-1)$-Kähler weighted with respect to K) on CY_4 are $\hat{\Omega}_4$, $D_a\hat{\Omega}_4$, $D_aD_b\hat{\Omega}_4$, $D_aD_bD_c\hat{\Omega}_4$ and $\overline{D}_{\overline{a}}D_bD_c\hat{\Omega}_4$.

As it can be realized by considering Eqs. (I.1)-(I.4) of Appendix I, $D_aD_b\hat{\Omega}_4$ cannot be expressed in terms of $D_a\hat{\Omega}_4$ (as instead it happens in the extremal BH case, see Eq. (3.4.1.19)), and all the independent, $(1,-1)$-Kähler-weighted 4-forms on the considered CY_4 are given by the $2h_{2,1}+2$ forms

$$\hat{\Omega}_4,\ D_0\hat{\Omega}_4,\ D_i\hat{\Omega}_4,\ D_0D_i\hat{\Omega}_4.\tag{4.1.3.1}$$

The third covariant derivatives of $\hat{\Omega}_4$ do not add any other independent 4-form, and so do all the other higher order covariant derivatives of $\hat{\Omega}_4$. Thus, the possible candidates along which one might

[17] For an elegant and detailed derivation of the Hodge-decomposition of $\mathfrak{F}_4$ using methods of algebraic geometry, see *e.g.* Sect. 2 of [85].

decompose the real flux 4-form $\mathfrak{F}_4$ are the 4-forms given by Eq. (4.1.3.1) and their complex conjugated $\overline{\hat{\Omega}}_4,\ \overline{D}_{\overline{0}}\overline{\hat{\Omega}}_4,\ \overline{D}_{\overline{i}}\overline{\hat{\Omega}}_4,\ \overline{D}_{\overline{0}}\overline{D}_{\overline{i}}\overline{\hat{\Omega}}_4$.

The *"intersections"* among the elements of the set of 4-forms $\hat{\Omega}_4$, $D_0\hat{\Omega}_4$, $D_i\hat{\Omega}_4$, $D_0D_i\hat{\Omega}_4$, $\overline{\hat{\Omega}}_4$, $\overline{D}_{\overline{0}}\overline{\hat{\Omega}}_4$, $\overline{D}_{\overline{i}}\overline{\hat{\Omega}}_4$ and $\overline{D}_{\overline{0}}\overline{D}_{\overline{i}}\overline{\hat{\Omega}}_4$ in generic local "curved" and in local "flat" coordinates of M are given in Appendix II. By using such results, the real, Kähler gauge-invariant 4-form $\mathfrak{F}_4$ can be thus Hodge-decomposed as follows ($\eta_1, ..., \eta_6 \in \mathbb{C}$)

$$\mathfrak{F}_4 = \left[\begin{array}{l} \eta_1\left(\int_{CY_4}\mathfrak{F}_4\wedge\hat{\Omega}_4\right)\overline{\hat{\Omega}}_4+\eta_2\delta^{A\overline{B}}\left(\int_{CY_4}\mathfrak{F}_4\wedge\left(D_A\hat{\Omega}_4\right)\right)\overline{D}_{\overline{B}}\overline{\hat{\Omega}}_4+ \\ +\eta_3\delta^{A\overline{B}}\left(\int_{CY_4}\mathfrak{F}_4\wedge D_{\underline{0}}D_A\hat{\Omega}_4\right)\overline{D}_{\underline{\overline{0}}}\overline{D}_{\overline{B}}\overline{\hat{\Omega}}_4+\eta_4\delta^{B\overline{A}}\left(\int_{CY_4}\mathfrak{F}_4\wedge\overline{D}_{\underline{\overline{0}}}\overline{D}_{\overline{A}}\overline{\hat{\Omega}}_4\right)D_{\underline{0}}D_B\hat{\Omega}_4+ \\ +\eta_5\delta^{B\overline{A}}\left(\int_{CY_4}\mathfrak{F}_4\wedge\left(\overline{D}_{\overline{A}}\overline{\hat{\Omega}}_4\right)\right)D_B\hat{\Omega}_4+\eta_6\left(\int_{CY_4}\mathfrak{F}_4\wedge\overline{\hat{\Omega}}_4\right)\hat{\Omega}_4 \end{array}\right] = \tag{4.1.3.2}$$

$$= \left[\begin{array}{l} \eta_1 Z\overline{\hat{\Omega}}_4+\eta_2\delta^{A\overline{B}}\left(D_AZ\right)\overline{D}_{\overline{B}}\overline{\hat{\Omega}}_4+\eta_3\delta^{A\overline{B}}\left(D_{\underline{0}}D_AZ\right)\overline{D}_{\underline{\overline{0}}}\overline{D}_{\overline{B}}\overline{\hat{\Omega}}_4+ \\ +\eta_4\delta^{B\overline{A}}\left(\overline{D}_{\underline{\overline{0}}}\overline{D}_{\overline{A}}\overline{Z}\right)D_{\underline{0}}D_B\hat{\Omega}_4+\eta_5\delta^{B\overline{A}}\left(\overline{D}_{\overline{A}}\overline{Z}\right)D_B\hat{\Omega}_4+\eta_6\overline{Z}\hat{\Omega}_4 \end{array}\right], \tag{4.1.3.3}$$

where Eq. (4.1.2.20) was used, also implying:

$$\begin{array}{l} \int_{CY_4}\mathfrak{F}_4\wedge D_a\hat{\Omega}_4 = D_aZ, \quad \int_{CY_4}\mathfrak{F}_4\wedge D_aD_b\hat{\Omega}_4 = D_aD_bZ; \\ \int_{CY_4}\mathfrak{F}_4\wedge D_A\hat{\Omega}_4 = D_AZ, \quad \int_{CY_4}\mathfrak{F}_4\wedge D_AD_B\hat{\Omega}_4 = D_AD_BZ. \end{array} \tag{4.1.3.4}$$

The r.h.s. of the Hodge-decomposition (4.1.3.3) is the most general Kähler gauge-invariant combination of *all* the possible ($(1,-1)$ and $(-1,1)$)-Kähler-weighted independent 4-forms for Type IIB on $CY_4 = \frac{CY_3\times T^2}{\mathbb{Z}_2}$. The reality condition $\overline{\mathfrak{F}}_4 = \mathfrak{F}_4$ implies $\eta_4 = \overline{\eta_3}$, $\eta_5 = \overline{\eta_2}$ and $\eta_6 = \overline{\eta_1}$. The (a priori) complex coefficients η_1, η_2 and η_3 can be determined by computing $\int_{CY_4}\mathfrak{F}_4\wedge\hat{\Omega}_4$, $\int_{CY_4}\mathfrak{F}_4\wedge D_A\hat{\Omega}_4$ and $\int_{CY_4}\mathfrak{F}_4\wedge D_{\underline{0}}D_A\hat{\Omega}_4$, and using the identity (4.1.3.3) and recalling Eqs. (4.1.2.20), (4.1.3.4) and the *"intersections"* in local "flat" coordinates (II.5)-(II.8). By doing so, one obtains:

$$\begin{array}{l} Z = \int_{CY_4}\mathfrak{F}_4\wedge\hat{\Omega}_4 = \eta_1 Z\int_{CY_4}\overline{\hat{\Omega}}_4\wedge\hat{\Omega}_4 = \eta_1 Z \Leftrightarrow \eta_1 = 1; \\ \\ D_CZ = \int_{CY_4}\mathfrak{F}_4\wedge D_C\hat{\Omega}_4 = \eta_2\delta^{A\overline{B}}\left(D_AZ\right)\int_{CY_4}\left(\overline{D}_{\overline{B}}\overline{\hat{\Omega}}_4\right)\wedge D_C\hat{\Omega}_4 = \\ = -\eta_2\delta^{A\overline{B}}D_AZ\delta_{C\overline{B}} = -\eta_2 D_CZ \Leftrightarrow \eta_2 = -1; \\ \\ D_{\underline{0}}D_CZ = \int_{CY_4}\mathfrak{F}_4\wedge D_{\underline{0}}D_C\hat{\Omega}_4 = \eta_3\delta^{A\overline{B}}\left(D_{\underline{0}}D_AZ\right)\int_{CY_4}\left(\overline{D}_{\underline{\overline{0}}}\overline{D}_{\overline{B}}\overline{\hat{\Omega}}_4\right)\wedge D_{\underline{0}}D_C\hat{\Omega}_4 = \\ = \eta_3\delta^{A\overline{B}}\delta_{C\overline{B}}D_{\underline{0}}D_AZ = \eta_3 D_{\underline{0}}D_CZ \Leftrightarrow \eta_3 = 1. \end{array} \tag{4.1.3.5}$$

Thus, the complete Hodge-decomposition of the real, Kähler gauge-invariant 4-form $\mathfrak{F}_4$ of Type IIB on

$\frac{CY_3\times T^2}{\mathbb{Z}_2}$ in generic local "flat" coordinates[18] in M reads

$$\mathfrak{F}_4 = 2Re\left[\overline{Z}\hat{\Omega}_4 - \left(\overline{D}^A\overline{Z}\right)D_A\hat{\Omega}_4 + \left(\overline{D}^0\overline{D}^A\overline{Z}\right)D_{\underline{0}}D_A\hat{\Omega}_4\right] = \tag{4.1.3.6}$$

$$= 2Re\left[\begin{array}{l}\overline{Z}\hat{\Omega}_1\wedge\hat{\Omega}_3 - \left(\overline{D}_{\overline{0}}\overline{Z}\right)\overline{\hat{\Omega}}_1\wedge\hat{\Omega}_3 - \left(\overline{D}^I\overline{Z}\right)\hat{\Omega}_1\wedge D_I\hat{\Omega}_3 + \\ + \left(\overline{D}^0\overline{D}^I\overline{Z}\right)\overline{\hat{\Omega}}_1\wedge D_I\hat{\Omega}_3\end{array}\right]. \tag{4.1.3.7}$$

4.2 $\mathcal{N}=1$, $d=4$ Effective Potential and "*Criticality Conditions*" *Approach*

The potential of $\mathcal{N}=1$, $d=4$ supergravity (from Type IIB on $\frac{CY_3\times T^2}{\mathbb{Z}_2}$), which acts as effective potential for the FV attractors, is given by Eq. (1.17), which we repeat here [81, 82]:

$$\begin{array}{l}V_{\mathcal{N}=1} = e^K\left[-3|W|^2 + g^{a\overline{b}}D_aW\overline{D}_{\overline{b}}\overline{W}\right] = -3|Z|^2 + g^{a\overline{b}}D_aZ\overline{D}_{\overline{b}}\overline{Z} = \\ = -3|Z|^2 - \left(\overline{t}^{\overline{0}} - t^0\right)^2|D_0Z|^2 + g^{i\overline{j}}\left(t^k,\overline{t}^{\overline{k}}\right)D_iZ\overline{D}_{\overline{j}}\overline{Z} \gtreqless 0.\end{array} \tag{4.2.1}$$

At a glance, the first difference between the "BH effective potential" V_{BH} given by Eq. (3.1.1) and the "FV effective potential" $V_{\mathcal{N}=1}$ given by Eq. (4.2.1) concerns their sign. Indeed, V_{BH} is positive-definite and it can be recognized as the first, quadratic invariant of SK geometry; through the Bekenstein-Hawking entropy-area formula, it is related to the classical entropy and to the area of the event horizon of the considered extremal (static, spherically symmetric, asymptotically flat) BH. On the other hand, $V_{\mathcal{N}=1}$ does not have a definite sign, and critical points of $V_{\mathcal{N}=1}$ can exist with $V_{\mathcal{N}=1} \gtreqless 0$:

1) $V_{\mathcal{N}=1}|_{\partial V_{\mathcal{N}=1}=0} > 0$ corresponds to De Sitter (dS) vacua;
2) $V_{\mathcal{N}=1}|_{\partial V_{\mathcal{N}=1}=0} = 0$ determines Minkowski vacua;
3) $V_{\mathcal{N}=1}|_{\partial V_{\mathcal{N}=1}=0} < 0$ corresponds to anti De Sitter (AdS) vacua.

By differentiating Eq. (4.2.1) with respect to the moduli and recalling Eqs. (4.1.2.19) and (4.1.2.25), one obtains the general criticality conditions of $V_{\mathcal{N}=1}$ ($\forall a = 0,1,...,h_{2,1}$):

$$\begin{array}{c}D_aV_{\mathcal{N}=1} = \partial_aV_{\mathcal{N}=1} = 0; \\ \Updownarrow \\ e^K\left[-3\overline{W}D_aW + g^{b\overline{c}}\left(D_aD_bW\right)\overline{D}_{\overline{c}}\overline{W} + g^{b\overline{c}}\left(D_bW\right)D_a\overline{D}_{\overline{c}}\overline{W}\right] = \\ = e^K\left[-2\overline{W}D_aW + g^{b\overline{c}}\left(D_aD_bW\right)\overline{D}_{\overline{c}}\overline{W}\right] = 0; \\ \Updownarrow \\ -2\overline{W}D_aW + g^{b\overline{c}}\left(D_aD_bW\right)\overline{D}_{\overline{c}}\overline{W} = 0,\end{array} \tag{4.2.2}$$

where, as in the case of extremal BH attractors in $\mathcal{N}=2$, $d=4$ supergravity, we assumed the Kähler potential to be regular, *i.e.* that $|K| < \infty$ globally in M (or *at least* at the critical points of $V_{\mathcal{N}=1}$).

[18] For the analogous expression in generic local "curved" coordinates in M, see Eqs. (III.1)-(III.4).

Eqs. (4.2.2) are the what one should rigorously refer to as the $\mathcal{N}=1$, $d=4$ FV AEs (in Type IIB on $\frac{CY_3\times T^2}{\mathbb{Z}_2}$). By recalling Eq. (4.1.1.12), they can finally be rewritten as

$$D_a V_{\mathcal{N}=1} \quad = \quad \partial_a V_{\mathcal{N}=1}=0 \Leftrightarrow -2\overline{W}D_aW+g^{0\overline{0}}\left(D_aD_0W\right)\overline{D}_{\overline{0}}\overline{W}+g^{i\overline{j}}\left(D_aD_iW\right)\overline{D}_{\overline{j}}\overline{W}=0,\forall a=0,1,...,h_{2,1}. \tag{4.2.3}$$

Let us specify such FV AEs for the two classes of (local "curved") indices; with some elaborations, one obtains:

$$\begin{aligned}&a=0 \text{ (axion-dilaton direction in } M):\\ &D_0V_{\mathcal{N}=1}=\partial_0V_{\mathcal{N}=1}=0\Leftrightarrow -2\overline{W}D_0W+g^{j\overline{k}}\left(D_0D_jW\right)\overline{D}_{\overline{k}}\overline{W}=0;\end{aligned} \tag{4.2.4}$$

$$\begin{gathered}a=i\ \in\{1,...,h_{2,1}\}\ \text{(CS directions in } M):\\ D_iV_{\mathcal{N}=1}=\partial_iV_{\mathcal{N}=1}=0\\ \Updownarrow\\ -2\overline{W}D_iW+g^{0\overline{0}}\left(D_iD_0W\right)\overline{D}_{\overline{0}}\overline{W}+g^{j\overline{k}}\left(D_iD_jW\right)\overline{D}_{\overline{k}}\overline{W}=0;\\ \Updownarrow\\ -2\overline{W}D_iW+e^{-2K_1}\left(D_0D_iW\right)\overline{D}_{\overline{0}}\overline{W}-e^{-K_1}g^{j\overline{k}}C_{ijl}g^{l\overline{m}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{m}}\overline{W}\right)\overline{D}_{\overline{k}}\overline{W}=0.\end{gathered} \tag{4.2.5}$$

Thus, despite the presence of the universal axion-dilaton direction in the $(h_{2,1}+1)$-dim. Kähler moduli space M, Eqs. (4.2.5) yields that the tensor C_{ijk}, defined in the $h_{2,1}$-dim. SK CS moduli space $\mathcal{M}_{CS}\subsetneq M$, still plays a key role. The FV AEs (4.2.4) and (4.2.5) of $\mathcal{N}=1$, $d=4$ supergravity from Type IIB on $\frac{CY_3\times T^2}{\mathbb{Z}_2}$ relate, at the critical points of the "FV effective potential" $V_{\mathcal{N}=1}$ (given by Eq. (4.2.1)), the $\mathcal{N}=1$, $d=4$ holomorphic superpotential W, the *supersymmetry order parameters* $D_iZ=e^{\frac{1}{2}K}D_iW$ and the *axino-dilatino-CS modulino mixings* $D_0D_iZ=e^{\frac{1}{2}K}D_0D_iW$, which is part of the $(h_{2,1}+1)\times(h_{2,1}+1)$ *modulino mass matrix* $\Lambda_{ab}\equiv D_aD_bZ=e^{\frac{1}{2}K}D_aD_bW$ (note that in the considered $\mathcal{N}=1$, $d=4$ framework the axino-dilatino and the $h_{2,1}$ CS modulinos play the role of the $n_V=h_{2,1}$ CS modulinos in the context of $\mathcal{N}=2$, $d=4$ supergravity from Type IIB on CY_3). It is worth pointing out that Λ_{ab} is part of the holomorphic/anti-holomorphic form of the $(2h_{2,1}+2)\times(2h_{2,1}+2)$ covariant Hessian of Z, which is nothing but the holomorphic/anti-holomorphic form of the scalar (axion-dilaton + CS moduli, in the stringy description as Type IIB on $\frac{CY_3\times T^2}{\mathbb{Z}_2}$) mass matrix.

The structure of the criticality conditions (4.2.3) and (4.2.4)-(4.2.5) suggests the classification of the critical points of $V_{\mathcal{N}=1}$ in two general classes:

I) The *supersymmetric (SUSY)* critical points of $V_{\mathcal{N}=1}$, determined by the differential constraints

$$D_aW=0,\forall a=0,1,...,h_{2,1}, \tag{4.2.6}$$

which directly solve the conditions (4.2.3) and (4.2.4)-(4.2.5). By substituting the SUSY FV constraints (4.2.6) into the expression (4.2.1) of $V_{\mathcal{N}=1}$, one obtains that SUSY dS critical points of $V_{\mathcal{N}=1}$ (*i.e.*,

independently on the stability, SUSY dS FV described by a classical FV Attractor Mechanism encoded by - the criticality conditions of - the potential $V_{\mathcal{N}=1}$) cannot exist, because

$$V_{\mathcal{N}=1,SUSY}=-3\left(e^{K}\left|W\right|^{2}\right)_{SUSY}=-3\left|Z\right|_{SUSY}^{2}\leqslant 0. \tag{4.2.7}$$

II) The *non-supersymmetric (non-SUSY)* critical points of $V_{\mathcal{N}=1}$, determined by the differential constraints

$$\left\{\begin{array}{l} D_{a}W\neq 0,\ (\textit{at least})\text{ for some } a\in\{0,1,...,h_{2,1}\}\,; \\ \\ \partial_{a}V_{\mathcal{N}=1}=0,\forall a=0,1,...,h_{2,1}. \end{array}\right. \tag{4.2.8}$$

The expression (4.2.1) of $V_{\mathcal{N}=1}$ suggests that *a priori* such critical points of $V_{\mathcal{N}=1}$ are of all possible species (dS, Minkowski, AdS).

4.3 Supersymmetric Flux Vacua Attractor Equations

In the present Subsection we will concentrate on the SUSY critical points of $V_{\mathcal{N}=1}$, determining the supersymmetric FV AEs in $\mathcal{N}=1$, $d=4$ supergravity from Type IIB on $\frac{CY_{3}\times T^{2}}{\mathbb{Z}_{2}}$. This can be achieved respectively by evaluating the Hodge identities (4.1.3.6)-(4.1.3.7) and (III.1)-(III.4) at the SUSY FV constraints (4.2.6).

The evaluation of the identities (4.1.3.6)-(4.1.3.7) and (III.1)-(III.4) along the constraints (4.2.6) respectively yields the supersymmetric FV AEs in $\mathcal{N}=1$, $d=4$ supergravity from Type IIB on $\frac{CY_{3}\times T^{2}}{\mathbb{Z}_{2}}$ in local "flat" coordinates[19]:

$$\begin{aligned}\mathfrak{F}_{4}&=2Re\left[\overline{Z}\hat{\Omega}_{4}+\delta^{A\overline{B}}\left(\overline{D}_{\underline{0}}\overline{D}_{\overline{B}}\overline{Z}\right)D_{\underline{0}}D_{A}\hat{\Omega}_{4}\right]_{SUSY}=\\&=2Re\left[\overline{Z}\hat{\Omega}_{1}\wedge\hat{\Omega}_{3}+\delta^{I\overline{J}}\left(\overline{D}_{\underline{0}}\overline{D}_{\overline{J}}\overline{Z}\right)\overline{\hat{\Omega}}_{1}\wedge D_{I}\hat{\Omega}_{3}\right]_{SUSY}=\\&=2e^{K_{1}+K_{3}}Re\left[\overline{W}\Omega_{1}\wedge\Omega_{3}+\delta^{I\overline{J}}\left(\overline{D}_{\underline{0}}\overline{D}_{\overline{J}}\overline{W}\right)\overline{\Omega}_{1}\wedge D_{I}\Omega_{3}\right]_{SUSY}.\end{aligned} \tag{4.3.1}$$

Notice that, as in Eqs. (4.3.1) as well as in the treatment below, the subscript "$SUSY$" denotes the evaluation at the SUSY FV constraints (4.2.6).

Furthermore, Eqs. (4.3.1) can be further elaborated by computing that

$$\begin{aligned}&\left(D_{\underline{0}}D_{J}W\right)_{SUSY}=\left(e_{J}^{j}\partial_{j}D_{\underline{0}}W\right)_{SUSY}=\left(e_{J}^{j}e_{\underline{0}}^{0}\partial_{j}D_{0}W\right)_{SUSY}=\left(e_{J}^{j}e_{\underline{0}}^{0}\partial_{0}D_{j}W\right)_{SUSY}=\\&=\left\{e_{J}^{j}\left(\overline{\tau}-\tau\right)\left[\partial_{j}\partial_{0}W+\tfrac{1}{2}\left(\partial_{j}K_{3}\right)\partial_{0}W\right]\right\}_{SUSY}=\\&=\left\{e_{J}^{j}\left(\overline{\tau}-\tau\right)\left[-q_{h|\Lambda}\partial_{j}X^{\Lambda}+p_{h}^{\Lambda}\partial_{j}F_{\Lambda}+\tfrac{1}{2}\frac{\left(\overline{X}^{\Sigma}\partial_{j}F_{\Sigma}-\overline{F}_{\Sigma}\partial_{j}X^{\Sigma}\right)}{\overline{X}^{\Delta}F_{\Delta}-\overline{F}_{\Delta}X^{\Delta}}\left(q_{h|\Xi}X^{\Xi}-p_{h}^{\Xi}F_{\Xi}\right)\right]\right\}_{SUSY}.\end{aligned} \tag{4.3.2}$$

The structure of the SUSY FV AEs (4.3.1) suggests the classification of the SUSY critical points of $V_{\mathcal{N}=1}$ in three general classes:

[19]For the analogous expression in generic local "curved" coordinates in M, see Eq. (III.5).

I) *Type "(3,0)"* SUSY FV, determined by the constraints (4.2.6) and by the further conditions

$$\left\{\begin{array}{l} W_{SUSY}\neq 0; \\ \qquad\qquad\left[g^{i\bar{j}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{W}\right)\overline{\Omega}_1\wedge D_i\Omega_3\right]_{SUSY}=0; \\ \qquad\qquad\qquad\qquad\Updownarrow \\ \forall\Lambda=1,...,h_{2,1}+1:\left\{\begin{array}{l}\left[g^{i\bar{j}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{W}\right)D_iX^{\Lambda}\right]_{SUSY}=0; \\ \left[g^{i\bar{j}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{W}\right)D_iF_{\Lambda}\right]_{SUSY}=0.\end{array}\right.\end{array}\right. \tag{4.3.3}$$

Because of

$$V_{\mathcal{N}=1,SUSY,(3,0)}=-3\left(e^{K}\left|W\right|^{2}\right)_{SUSY,(3,0)}<0, \tag{4.3.4}$$

the class *"(3,0)"* of SUSY FV is composed only by AdS FV. In this case, the SUSY FV AEs read as follows:

$$\begin{array}{l}\mathfrak{F}_4=2Re\left[\overline{Z}\hat{\Omega}_4\right]_{SUSY,(3,0)}= \\ =2Re\left[\overline{Z}\hat{\Omega}_1\wedge\hat{\Omega}_3\right]_{SUSY,(3,0)}=2\left(e^{K_1+K_3}\right)_{SUSY,(3,0)}Re\left[\overline{W}\Omega_1\wedge\Omega_3\right]_{SUSY,(3,0)};\end{array} \tag{4.3.5}$$

Notice that for such a class of SUSY FV the condition of consistence of Eqs. (4.3.5) is $W_{SUSY}\neq 0$. In other words, Minkowski ($V_{\mathcal{N}=1}=0$) SUSY FV satisfying the constraints (4.2.6) and

$$\left\{\begin{array}{l} W_{SUSY}=0; \\ \qquad\qquad\left[g^{i\bar{j}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{W}\right)\overline{\Omega}_1\wedge D_i\Omega_3\right]_{SUSY}=0; \\ \qquad\qquad\qquad\qquad\Updownarrow \\ \forall\Lambda=1,...,h_{2,1}+1:\left\{\begin{array}{l}\left[g^{i\bar{j}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{W}\right)D_iX^{\Lambda}\right]_{SUSY}=0; \\ \left[g^{i\bar{j}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{W}\right)D_iF_{\Lambda}\right]_{SUSY}=0.\end{array}\right.\end{array}\right. \tag{4.3.6}$$

are not described by the classical FV Attractor Mechanism encoded by the SUSY FV AEs (4.3.1). Indeed, such Eqs., when evaluated along the constraints (4.3.6) simply return *all* (RR and NSNS) vanishing fluxes.

It is worth pointing out that, beside the substitution of "Im" with "Re" and the doubling of the vector dimension due to the $SL(2,\mathbb{R})$-doublet of RR and NSNS (3-form) fluxes, the *"(3,0)"* SUSY FV AEs (4.3.5) are very close to the SUSY extremal BH AEs in $\mathcal{N}=2$, $d=4$ supergravity, given by Eqs. (3.3.2.1)-(3.3.2.2).

II) *Type "(2,1)"* SUSY FV, determined by the constraints (4.2.6) and by the further conditions

$$\left\{ \begin{array}{l} W_{SUSY}=0; \\ \qquad \left[g^{i\overline{j}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{W}\right) \overline{\Omega}_{1}\wedge D_{i}\Omega_{3}\right] _{SUSY}\neq 0; \\ \qquad \Updownarrow \\ (\textit{at least})\text{ for some }\Lambda \in \left\{ 1,...,h_{2,1}+1\right\} :\left\{ \begin{array}{l} \left[g^{i\overline{j}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{W}\right) D_{i}X^{\Lambda }\right] _{SUSY}\neq 0; \\ \text{and/or} \\ \left[g^{i\overline{j}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{W}\right) D_{i}F_{\Lambda }\right] _{SUSY}\neq 0. \end{array} \right. \end{array} \right. \tag{4.3.7}$$

Because of

$$V_{\mathcal{N}=1,SUSY,(2,1)}=-3\left(e^{K}\left| W\right| ^{2}\right) _{SUSY,(2,1)}=0, \tag{4.3.8}$$

the class *"(2,1)"* of SUSY FV is composed only by Minkowski FV. In this case, the SUSY FV AEs read as follows:

$$\begin{array}{l} \mathfrak{F}_{4}=\left[\delta ^{A\overline{B}}\left(D_{\underline{0}}D_{A}Z\right) \overline{D}_{\overline{\underline{0}}}\overline{D}_{\overline{B}}\overline{\hat{\Omega}}_{4}+\delta ^{B\overline{A}}\left(\overline{D}_{\overline{\underline{0}}}\overline{D}_{\overline{A}}\overline{Z}\right) D_{\underline{0}}D_{B}\hat{\Omega}_{4}\right] _{SUSY,(2,1)}= \\ =2Re\left[\left(t^{0}-\overline{t}^{\overline{0}}\right) g^{i\overline{j}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{Z}\right) \overline{\hat{\Omega}}_{1}\wedge D_{i}\hat{\Omega}_{3}\right] _{SUSY,(2,1)}= \\ =2Re\left[\overline{e}_{\overline{\underline{0}}}^{\overline{0}}e_{I}^{i}\overline{e}_{\overline{J}}^{\overline{j}}\delta ^{I\overline{J}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{Z}\right) \overline{\hat{\Omega}}_{1}\wedge D_{i}\hat{\Omega}_{3}\right] _{SUSY,(2,1)}= \\ =2\left(e^{K_{1}+K_{3}}\right) _{SUSY,(2,1)}Re\left[\overline{e}_{\overline{\underline{0}}}^{\overline{0}}e_{I}^{i}\overline{e}_{\overline{J}}^{\overline{j}}\delta ^{I\overline{J}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{W}\right) \overline{\Omega}_{1}\wedge D_{i}\Omega_{3}\right] _{SUSY,(2,1)}. \end{array} \tag{4.3.9}$$

It is worth observing that the SUSY FV constraints (4.2.6) and the condition $W_{SUSY,(2,1)}=0$ imply

$$\begin{array}{c} \left(\partial_{a}W\right) _{SUSY,(2,1)}=0,\forall a=0,1,...,h_{2,1}; \\ \Updownarrow \\ \left\{ \begin{array}{l} \left(\partial_{0}W\right) _{SUSY,(2,1)}=\left(-q_{h|\Lambda }X^{\Lambda }+p_{h}^{\Lambda }F_{\Lambda }\right) _{SUSY,(2,1)}=0; \\ \forall i=1,...,h_{2,1}:\left\{ \begin{array}{l} \left(\partial_{i}W\right) _{SUSY,(2,1)}=\left[q_{f|\Lambda }\partial_{i}X^{\Lambda }-p_{f}^{\Lambda }\partial_{i}F_{\Lambda }-\tau \left(q_{h|\Lambda }\partial_{i}X^{\Lambda }-p_{h}^{\Lambda }\partial_{i}F_{\Lambda }\right) \right] _{SUSY,(2,1)}=0; \\ \qquad \Updownarrow \\ \left(q_{h|\Lambda }\partial_{i}X^{\Lambda }-p_{h}^{\Lambda }\partial_{i}F_{\Lambda }\right) _{SUSY,(2,1)}=\left[\frac{1}{\tau }\left(q_{f|\Lambda }\partial_{i}X^{\Lambda }-p_{f}^{\Lambda }\partial_{i}F_{\Lambda }\right) \right] _{SUSY,(2,1)}, \end{array} \right. \end{array} \right. \end{array} \tag{4.3.10}$$

where we used the fact that $\tau \neq 0$ is a necessary (but not sufficient) condition for the (assumed) regularity of K_1 in $\mathcal{M}_{t^0 \equiv \tau} \subsetneq M$ (or at least in the considered critical points of $V_{\mathcal{N}=1}$). Thus, Eq. (4.3.2) can be

further elaborated as follows:

$$
\begin{aligned}
&\left(D_{\underline{0}}D_JW\right)_{SUSY,(2,1)} = \left(e_J^j\partial_j D_{\underline{0}}W\right)_{SUSY,(2,1)} = \\
&= \left(e_J^j e_{\underline{0}}^0\partial_j\partial_0 W\right)_{SUSY,(2,1)} = \left(e_J^j e_{\underline{0}}^0\partial_0\partial_j W\right)_{SUSY,(2,1)} = \\
&= \left[e_J^j\left(\overline{\tau}-\tau\right)\partial_j\partial_0 W\right]_{SUSY,(2,1)} = \left\{e_J^j\left(\overline{\tau}-\tau\right)\left(-q_{h|\Lambda}\partial_j X^\Lambda + p_h^\Lambda\partial_j F_\Lambda\right)\right\}_{SUSY,(2,1)} = \\
&= \left\{-e_J^j\tfrac{\left(\overline{\tau}-\tau\right)}{\tau}\left(q_{f|\Lambda}\partial_j X^\Lambda - p_f^\Lambda\partial_j F_\Lambda\right)\right\}_{SUSY,(2,1)},
\end{aligned}
\tag{4.3.11}
$$

where in the last line we used Eq. (4.3.10). Furthermore, by using Eq. (4.3.10) with some elaborations, one obtains that

$$
\left.\begin{array}{l}
W_{SUSY,(2,1)} = 0 \\
\\
\left(\partial_0 W\right)_{SUSY,(2,1)} = 0
\end{array}\right\} \Rightarrow \left(q_{f|\Lambda}X^\Lambda - p_f^\Lambda F_\Lambda\right)_{SUSY,(2,1)} = 0,
\tag{4.3.12}
$$

and therefore at the class *"(2,1)"* of SUSY critical points of $V_{\mathcal{N}=1}$ the *"RR sector"* $q_{f|\Lambda}X^\Lambda - p_f^\Lambda F_\Lambda$ and the "*NSNS sector*" $-\left(q_{h|\Lambda}X^\Lambda - p_h^\Lambda F_\Lambda\right)$ of the holomorphic superpotential W vanish separately.

By looking at the *"(2,1)"* SUSY FV AEs (4.3.9), it is interesting to note that *"(2,1)"* SUSY FV do not have a counterpart in the theory of extremal BH attractors in $\mathcal{N}=2$, $d=4$ supergravity. Indeed, as implied by the SUSY extremal BH AEs (3.3.2.1)-(3.3.2.2), the classical extremal BH Attractor Mechanism in $\mathcal{N}=2$, $d=4$ supergravity is not consistent with SUSY critical points of V_{BH} also having $W=0$, and thus determining $V_{BH}=0$. In such a case, the SUSY extremal BH AEs (3.3.2.1)-(3.3.2.2) simply yield *all* (magnetic and electric) BH charges vanishing.

Contrarily to the extremal BH attractors in $\mathcal{N}=2$, $d=4$ supergravity, and as yielded by the *"(2,1)"* SUSY FV AEs (4.3.9), the classical FV Attractor Mechanism allows for stabilization of (axion-dilaton + CS) moduli in the SUSY case with vanishing *gravitino mass* $Z_{USY} = \left(e^K W\right)_{SUSY} = 0$.

III) *Type "(3,0)+(2,1)"* SUSY FV, determined by the constraints (4.2.6) and by the further conditions

$$
\left\{\begin{array}{l}
W_{SUSY} \neq 0; \\
\qquad\qquad \left[g^{i\overline{j}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{W}\right)\overline{\Omega}_1 \wedge D_i\Omega_3\right]_{SUSY} \neq 0; \\
\qquad\qquad \Updownarrow \\
(\textit{at least}) \text{ for some } \Lambda \in \{1,...,h_{2,1}+1\}: \left\{\begin{array}{l}
\left[g^{i\overline{j}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{W}\right)D_i X^\Lambda\right]_{SUSY} \neq 0; \\
\text{and/or} \\
\left[g^{i\overline{j}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{W}\right)D_i F_\Lambda\right]_{SUSY} \neq 0.
\end{array}\right.
\end{array}\right.
\tag{4.3.13}
$$

Because of

$$
V_{\mathcal{N}=1,SUSY,(3,0)+(2,1)} = -3\left(e^K|W|^2\right)_{SUSY,(3,0)+(2,1)} < 0,
\tag{4.3.14}
$$

the class *"(3,0)+(2,1)"* of SUSY FV is composed only by AdS FV. For such a class of SUSY FV the FV AEs are simply given by Eqs. (4.3.1) (which can be further elaborated by considering Eq. (4.3.2)), constrained by Eqs. (4.3.13).

In [11] examples of SUSY FV of all the classes considered above (*"(3,0)"*, *"(2,1)"*, and *"(3,0)+(2,1)"*) have been explicitly checked to satisfy the corresponding SUSY FV AEs of $\mathcal{N} = 1$, $d = 4$ supergravity from Type IIB on $\frac{CY_3 \times T_2}{\mathbb{Z}_2}$, in a model with $h_{2,1} = 1$, where CY_3 is the so-called Fermat *sixtic* hypersurface.

5 Some Recent Developments on Extremal Black Hole Attractors

In these lectures we have described the general theory of attractors for a generic $\mathcal{N} = 2$, $d = 4$ SK geometry, both in the supergravity language and in terms of Type IIB superstrings compactified on Calabi-Yau threefolds. We have then described, in a similar way, the Attractor Mechanism arising in $\mathcal{N} = 1$, $d = 4$ flux vacua, focussing on the case of (the F-theory limit of) compactifications of Type IIB on Calabi-Yau orientifolds (see also [110] for an extension to the landscape of non-Kähler vacua emerging in the flux compactifications of heterotic superstrings).

In the last years, more results have been obtained for the non-BPS extremal $d = 4$ BH attractors, especially with regard to symmetric $\mathcal{N} = 2$ SK geometries and to $\mathcal{N} > 2$ extended theories.

The classification of the charge orbits of the U-duality [111] groups supporting attractors with non-vanishing entropy was performed in [70] (see also [19]) and [21], respectively for $\mathcal{N} = 8$ and $\mathcal{N} = 2$ symmetric supergravities, whereas the corresponding moduli spaces were found and studied respectively in [36] and [33]. Furthermore, the classification of attractors for $\mathcal{N} = 3, 4$ (along with the corresponding maximal compact symmetries) was performed in [74]. Notice that the $\mathcal{N} = 6$ theory has the same attractors, orbits and related moduli spaces of the quaternionic magic $\mathcal{N} = 2$ model [21, 112].

For the sake of completeness we report here the charge orbits and the moduli space[20] of attractors for all $3 \leqslant \mathcal{N} \leqslant 8$, $d = 4$ supergravities (for the treatment of extremal BHs in such theories, see *e.g.* [113, 104, 114, 115, 116]), including the cases $\mathcal{N} = 3, 4, 5$, not exhaustively discussed in literature.

All $d = 4$ theories with $\mathcal{N}$ even can be uplifted to $d = 5$, and their U-duality group admits a unique quartic invariant (see *e.g.* [117]). All such supergravities have a non-BPS attractor solution whose moduli space coincide with the $d = 5$ real scalar manifold. This is the non-BPS solution with non-vanishing central charge matrix Z_{AB} (A, $B = 1, ..., \mathcal{N}$), which breaks the $d = 4$ $\mathcal{R}$-symmetry to the $d = 5$ $\mathcal{R}$-symmetry. Since the cases $\mathcal{N} = 6, 8$ have been treated in [19, 21, 74, 33, 36], let us now consider the case $\mathcal{N} = 4$; as previously mentioned, its attractors with non-vanishing entropy (and the corresponding maximal compact symmetries) have been classified in [74]. The non-BPS attractor with $Z_{AB} \neq 0$ breaks the $d = 4$ $\mathcal{R}$-symmetry $SU(4) \sim SO(6)$ down to the $d = 5$ $\mathcal{R}$-symmetry $USp(4) \sim SO(5)$, and the maximal compact symmetry exhibited by the solution is $USp(4) \otimes SO(n-1)$, where n denotes the number of matter multiplets coupled to the supergravity one. The other non-BPS attractor solution of $\mathcal{N} = 4$, $d = 4$ supergravity has $Z_{AB} = 0$; thus, the $d = 4$ $\mathcal{R}$-symmetry $SU(4)$ is unbroken, and the corresponding maximal compact symmetry is $SU(4) \otimes SO(n-2)$.

On the other hand, the $d = 4$ theories with $\mathcal{N}$ odd ($= 3, 5$) cannot be uplifted to $d = 5$, and their U-duality group admits a unique quadratic invariant (see *e.g.* [117]). The case $\mathcal{N} = 3$ admits only

[20] The scalar manifolds of $3 \leqslant \mathcal{N} \leqslant 8$, $d = 4$ supergravities can be found *e.g.* in [74].

$\frac{1}{3}$-BPS and non-BPS $Z_{AB}=0$ attractors with non-vanishing entropy; notice that such a result is similar to the one obtained for the $\mathcal{N}=2$ symmetric sequence of SK manifolds based on quadratic holomorphic prepotential (see [21], [36] and Refs. therein), and it is ultimately due to the aforementioned fact that the $\mathcal{N}=3$, $d=4$ U-duality group $SU(3,n)$ has a unique quadratic (rather than quartic) invariant. $\mathcal{N}=5$ is peculiar, as discussed in [74], in such a case only the $\frac{1}{5}$-BPS attractor has non-vanishing entropy (this solution splits in BPS and non-BPS $Z=0$ ones when performing the $\mathcal{N}=5\rightarrow\mathcal{N}=2$ truncation of the theory [74]).

By knowing the real, symplectic representation R (with $dim_{\mathbb{R}}=\mathfrak{r}$) of the U-duality group G in which the charge vector Q sits, the orbits of R supporting attractors with non-vanishing entropy can be computed; their dimension is always $\mathfrak{r}-1$, because they are defined by a fixed, non-vanishing value of the unique U-invariant of the theory. For ($\mathcal{N}=2$ symmetric and) $3\leqslant\mathcal{N}\leqslant 8$, $d=4$ supergravities such orbits are homogeneous symmetric manifolds of the form $\frac{G}{\mathfrak{H}}$ ($\mathfrak{H}=\mathcal{H},\widehat{\mathcal{H}},\widetilde{\mathcal{H}}$ respectively for BPS, non-BPS $Z_{AB}\neq 0$ and non-BPS $Z_{AB}=0$); the corresponding moduli space is given by the symmetric manifold $\frac{\mathfrak{H}}{h}$, where h ($=\mathfrak{h},\widehat{\mathfrak{h}},\widetilde{\mathfrak{h}}$, respectively) is the maximal compact subgroup of $\mathfrak{H}$. It is worth remarking that the ($\frac{1}{\mathcal{N}}$-)BPS moduli spaces of $3\leqslant\mathcal{N}\leqslant 8$, $d=4$ supergravities all are quaternionic Kähler manifolds; such a geometrical property can be understood by noticing that in the supersymmetry reduction down to $\mathcal{N}=2$ such spaces are spanned by the hypermultiplets' scalar degrees of freedom [117, 33].

Following [117] and [74], the relation among the signs of the U-invariant I_2 (quadratic in charges) or I_4 (quartic in charges) of the considered supergravity and the various BH charge orbits is (for the $\mathcal{N}=8$ case see also [118, 72, 70, 119, 19]):

$$\begin{array}{ll}
\mathcal{N}=3: & \left\{\begin{array}{l}\frac{1}{3}-BPS: I_2>0;\\ non-BPS, Z_{AB}=0: I_2<0;\end{array}\right.\\[2ex]
\mathcal{N}=4: & \left\{\begin{array}{l}\frac{1}{4}-BPS: I_4>0;\\ non-BPS, Z_{AB}\neq 0: I_4<0;\\ non-BPS, Z_{AB}=0: I_4>0;\end{array}\right.\\[2ex]
\mathcal{N}=5: & \frac{1}{5}-BPS: I_2\gtrless 0 \textit{(sign does not matter)};\\[2ex]
\mathcal{N}=6: & \left\{\begin{array}{l}\frac{1}{6}-BPS: I_4>0;\\ non-BPS, Z_{AB}\neq 0: I_4<0;\\ non-BPS, Z_{AB}=0: I_4>0;\end{array}\right.\\[2ex]
\mathcal{N}=8: & \left\{\begin{array}{l}\frac{1}{8}-BPS: I_4>0;\\ non-BPS, Z_{AB}\neq 0: I_4<0.\end{array}\right.
\end{array} \tag{5.1}$$

In Tables 1 and 2 we respectively list all charge orbits supporting extremal BH attractors with non-vanishing classical Bekenstein-Hawking [67] entropy (*i.e.* corresponding to the so-called "large" BHs) and their corresponding moduli spaces for $3\leqslant\mathcal{N}\leqslant 8$, $d=4$ supergravities.

Some of the above results hold also for a generic (non-symmetric nor eventually homogeneous) $\mathcal{N}=2$, $d=4$ SK geometry based on a cubic holomorphic prepotential (usually named SK d-geometry [93]). For instance, for any SK d-geometry of $\mathcal{N}=2$, $d=4$ supergravity coupled to n Abelian vector multiplets, the so-called $D0$-$D6$ BH charge configuration supports only non-BPS $Z\neq 0$ attractors, whose moduli space is $(n-1)$-dimensional, and it is given by the corresponding $\mathcal{N}=2$, $d=5$ scalar manifold, endowed

with real special geometry [34]. It is worth pointing out here that the existence of $n-1$ massless modes of the $2n \times 2n$ (real from of the) Hessian matrix of the BH effective potential V_{BH} at its non-BPS $Z \neq 0$ critical points was shown in [10] to hold in any SK d-geometry of $\mathcal{N}=2$, $d=4$ supergravity coupled to n Abelian vector multiplets. However, the issue of the stability of the non-BPS $Z \neq 0$ critical points of V_{BH} (as well as of the non-BPS $Z=0$ ones) in non-homogeneous SK d-geometry has not been thoroughly investigated so far[21].

Let us finally remark that it is also possible to relate the *flat* directions of non-BPS attractor solutions in $\mathcal{N}=2$, $d=4$ symmetric supergravities with the *flat* directions of ($\frac{1}{\mathcal{N}}$-)BPS and non-BPS attractors in $\mathcal{N}>2$, $d=4$ theories [33]. Moreover, the moduli spaces of extremal BH attractors with non-vanishing entropy in supergravity theories in $d=5$ and $d=6$ have been found and their relations with the corresponding Attractor Eqs. in $d=4$ studied in [40] and [49].

[21] The case of homogeneous non-symmetric SK d-geometry has been studied in [28].

	$\frac{1}{\mathcal{N}}$-BPS orbits $\frac{G}{\mathcal{H}}$	non-BPS, $Z_{AB} \neq 0$ orbits $\frac{G}{\widehat{\mathcal{H}}}$	non-BPS, $Z_{AB} = 0$ orbits $\frac{G}{\widetilde{\mathcal{H}}}$
$\mathcal{N} = 3$	$\frac{SU(3,n)}{SU(2,n)}$	$-$	$\frac{SU(3,n)}{SU(3,n-1)}$
$\mathcal{N} = 4$	$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(6,n)}{SO(4,n)}$	$\frac{SU(1,1)}{SO(1,1)} \otimes \frac{SO(6,n)}{SO(5,n-1)}$	$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(6,n)}{SO(6,n-2)}$
$\mathcal{N} = 5$	$\frac{SU(1,5)}{SU(3)\otimes SU(2,1)}$	$-$	$-$
$\mathcal{N} = 6$	$\frac{SO^*(12)}{SU(4,2)}$	$\frac{SO^*(12)}{SU^*(6)}$	$\frac{SO^*(12)}{SU(6)}$
$\mathcal{N} = 8$	$\frac{E_{7(7)}}{E_{6(2)}}$	$\frac{E_{7(7)}}{E_{6(6)}}$	$-$

Table 1: **Charge orbits of the real, symplectic R representation of the U-duality group G supporting BH attractors with non-vanishing entropy in $3 \leqslant \mathcal{N} \leqslant 8$, $d = 4$ supergravities**

	$\frac{1}{\mathcal{N}}$-BPS moduli space $\frac{\mathcal{H}}{\mathfrak{h}}$	non-BPS, $Z_{AB} \neq 0$ moduli space $\frac{\widehat{\mathcal{H}}}{\widehat{\mathfrak{h}}}$	non-BPS, $Z_{AB} = 0$ moduli space $\frac{\widetilde{\mathcal{H}}}{\widetilde{\mathfrak{h}}}$
$\mathcal{N} = 3$	$\frac{SU(2,n)}{SU(2)\otimes SU(n)\otimes U(1)}$	$-$	$\frac{SU(3,n-1)}{SU(3)\otimes SU(n-1)\otimes U(1)}$
$\mathcal{N} = 4$	$\frac{SO(4,n)}{SO(4)\otimes SO(n)}$	$SO(1,1) \otimes \frac{SO(5,n-1)}{SO(5)\otimes SO(n-1)}$	$\frac{SO(6,n-2)}{SO(6)\otimes SO(n-2)}$
$\mathcal{N} = 5$	$\frac{SU(2,1)}{SU(2)\otimes U(1)}$	$-$	$-$
$\mathcal{N} = 6$	$\frac{SU(4,2)}{SU(4)\otimes SU(2)\otimes U(1)}$	$\frac{SU^*(6)}{USp(6)}$	$-$
$\mathcal{N} = 8$	$\frac{E_{6(2)}}{SU(6)\otimes SU(2)}$	$\frac{E_{6(6)}}{USp(8)}$	$-$

Table 2: **Moduli spaces of BH attractors with non-vanishing entropy in $3 \leqslant \mathcal{N} \leqslant 8$, $d = 4$ supergravities ($\mathfrak{h}$, $\widehat{\mathfrak{h}}$ and $\widetilde{\mathfrak{h}}$ are maximal compact subgroups of $\mathcal{H}$, $\widehat{\mathcal{H}}$ and $\widetilde{\mathcal{H}}$, respectively)**

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A Appendix I

Up to the third order of covariant differentiation included, the possible independent 4-forms ($(1,-1)$-Kähler weighted with respect to K) on CY_4 beside $\hat{\Omega}_4$ are:

$$D_a\hat{\Omega}_4 \quad :$$

$$\left\{\begin{array}{l}
a=0: \\
D_0\hat{\Omega}_4=\left(D_0\hat{\Omega}_1\right)\wedge\hat{\Omega}_3=ie^{K_1}\overline{\hat{\Omega}}_1\wedge\hat{\Omega}_3=\left(\overline{t}^0-t^0\right)^{-1}\overline{\hat{\Omega}}_1\wedge\hat{\Omega}_3; \\
\\
a=i: \\
D_i\hat{\Omega}_4=\hat{\Omega}_1\wedge D_i\hat{\Omega}_3=e^{\frac{1}{2}K_3}\hat{\Omega}_1\wedge D_i\Omega_3=e^{\frac{1}{2}K_3}\hat{\Omega}_1\wedge\left[\partial_i\Omega_3+\left(\partial_iK_3\right)\Omega_3\right]= \\
=\frac{1}{\sqrt{i\left(\overline{X}^{\Delta}F_{\Delta}-X^{\Delta}\overline{F}_{\Delta}\right)}}\hat{\Omega}_1\wedge\left\{\begin{array}{l}
\left[\partial_iX^{\Lambda}-\frac{\left(\overline{X}^{\Sigma}\partial_iF_{\Sigma}-\left(\partial_iX^{\Sigma}\right)\overline{F}_{\Sigma}\right)}{\overline{X}^{\Xi}F_{\Xi}-X^{\Xi}\overline{F}_{\Xi}}X^{\Lambda}\right]\alpha_{\Lambda}+ \\
-\left[\partial_iF_{\Lambda}-\frac{\left(\overline{X}^{\Sigma}\partial_iF_{\Sigma}-\left(\partial_iX^{\Sigma}\right)\overline{F}_{\Sigma}\right)}{\overline{X}^{\Xi}F_{\Xi}-X^{\Xi}\overline{F}_{\Xi}}F_{\Lambda}\right]\beta^{\Lambda}
\end{array}\right\}.
\end{array}\right. \tag{I.1}$$

$$D_aD_b\hat{\Omega}_4=D_{(a}D_{b)}\hat{\Omega}_4:$$

$$\left\{\begin{array}{l}
(a,b)=(0,0): D_0D_0\hat{\Omega}_4=0; \\
\\
(a,b)=(0,i): \\
D_0D_i\hat{\Omega}_4=D_0\hat{\Omega}_1\wedge D_i\hat{\Omega}_3= \\
=\frac{1}{\left(\overline{t}^0-t^0\right)\sqrt{i\left(\overline{X}^{\Delta}F_{\Delta}-X^{\Delta}\overline{F}_{\Delta}\right)}}\overline{\hat{\Omega}}_1\wedge\left\{\begin{array}{l}
\left[\partial_iX^{\Lambda}-\frac{\left(\overline{X}^{\Sigma}\partial_iF_{\Sigma}-\left(\partial_iX^{\Sigma}\right)\overline{F}_{\Sigma}\right)}{\overline{X}^{\Xi}F_{\Xi}-X^{\Xi}\overline{F}_{\Xi}}X^{\Lambda}\right]\alpha_{\Lambda}+ \\
-\left[\partial_iF_{\Lambda}-\frac{\left(\overline{X}^{\Sigma}\partial_iF_{\Sigma}-\left(\partial_iX^{\Sigma}\right)\overline{F}_{\Sigma}\right)}{\overline{X}^{\Xi}F_{\Xi}-X^{\Xi}\overline{F}_{\Xi}}F_{\Lambda}\right]\beta^{\Lambda}
\end{array}\right\}; \\
\\
(a,b)=(i,j): \\
D_iD_j\hat{\Omega}_4=\hat{\Omega}_1\wedge D_iD_j\hat{\Omega}_3=iC_{ijk}g^{k\overline{l}}\hat{\Omega}_1\wedge\overline{D}_{\overline{l}}\overline{\hat{\Omega}}_3= \\
=-e^{-K_1}C_{ijk}g^{k\overline{l}}\overline{D}_{\overline{0}}\overline{\hat{\Omega}}_1\wedge\overline{D}_{\overline{l}}\overline{\hat{\Omega}}_3=-e^{-K_1}C_{ijk}g^{k\overline{l}}\overline{D}_{\overline{0}}\overline{D}_{\overline{l}}\overline{\hat{\Omega}}_4= \\
=-e^{-K_1}C_{ijk}g^{k\overline{l}}\overline{D}_{\overline{l}}\overline{D}_{\overline{0}}\overline{\hat{\Omega}}_4=-i\left(\overline{t}^0-t^0\right)C_{ijk}g^{k\overline{l}}\overline{D}_{\overline{l}}\overline{D}_{\overline{0}}\overline{\hat{\Omega}}_4.
\end{array}\right. \tag{I.2}$$

$D_a D_b D_c \hat{\Omega}_4 = D_{(a} D_b D_{c)} \hat{\Omega}_4 :$

$$
\begin{cases}
(a,b,c) = (0,0,0) : D_0 D_0 D_0 \hat{\Omega}_4 = 0; \\
(a,b,c) = (0,0,i) : D_0 D_0 D_i \hat{\Omega}_4 = 0; \\
(a,b,c) = (0,i,j) : \begin{cases} D_0 D_i D_j \hat{\Omega}_4 = D_0 \hat{\Omega}_1 \wedge D_i D_j \hat{\Omega}_3 = \\ = -e^{K_1} C_{ijk} g^{k\bar{l}} \overline{\hat{\Omega}}_1 \wedge \overline{D}_{\bar{l}} \overline{\hat{\Omega}}_3 = \\ = -e^{K_1} C_{ijk} g^{k\bar{l}} \overline{D}_{\bar{l}} \overline{\hat{\Omega}}_4 = i \left(\overline{t}^{\overline{0}} - t^0 \right)^{-1} C_{ijk} g^{k\bar{l}} \overline{D}_{\bar{l}} \overline{\hat{\Omega}}_4 ; \end{cases} \\
(a,b,c) = (i,j,k) : \begin{cases} D_i D_j D_k \hat{\Omega}_4 = -i \left(\overline{t}^{\overline{0}} - t^0 \right) \left(D_i C_{jkl} \right) g^{l\overline{m}} \overline{D}_{\overline{m}} \overline{D}_{\overline{0}} \overline{\hat{\Omega}}_4 + \\ -i \left(\overline{t}^{\overline{0}} - t^0 \right) C_{jkl} g^{l\overline{m}} D_i \overline{D}_{\overline{m}} \overline{D}_{\overline{0}} \overline{\hat{\Omega}}_4 = \\ = -i \left(\overline{t}^{\overline{0}} - t^0 \right) \left(D_i C_{jkl} \right) g^{l\overline{m}} \overline{D}_{\overline{0}} \overline{D}_{\overline{m}} \overline{\hat{\Omega}}_4 + \\ -i \left(\overline{t}^{\overline{0}} - t^0 \right) C_{jkl} g^{l\overline{m}} \left(\overline{D}_{\overline{0}} \overline{\hat{\Omega}}_1 \right) \wedge D_i \overline{D}_{\overline{m}} \overline{\hat{\Omega}}_3 = \\ = -i \left(\overline{t}^{\overline{0}} - t^0 \right) \left(D_i C_{jkl} \right) g^{l\overline{m}} \overline{D}_{\overline{0}} \overline{D}_{\overline{m}} \overline{\hat{\Omega}}_4 - i \left(\overline{t}^{\overline{0}} - t^0 \right) C_{ijk} \overline{D}_{\overline{0}} \overline{\hat{\Omega}}_4 . \end{cases}
\end{cases}
\tag{I.3}
$$

$$\overline{D}_{\overline{a}}D_bD_c\hat{\Omega}_4 = \overline{D}_{\overline{a}}D_{(b}D_{c)}\hat{\Omega}_4 :$$

$$\left\{\begin{array}{l}
(\overline{a},b,c)=\left(\overline{0},0,0\right):\overline{D}_{\overline{0}}D_0D_0\hat{\Omega}_4=0;\\
\\
(\overline{a},b,c)=\left(\overline{0},0,i\right):\overline{D}_{\overline{0}}D_0D_i\hat{\Omega}_4=g_{0\overline{0}}D_i\hat{\Omega}_4=-\left(\overline{t}^{\overline{0}}-t^0\right)^{-2}D_i\hat{\Omega}_4=e^{2K_1}D_i\hat{\Omega}_4;\\
\\
(\overline{a},b,c)=\left(\overline{0},i,j\right):\overline{D}_{\overline{0}}D_iD_j\hat{\Omega}_4=0;\\
\\
(\overline{a},b,c)=\left(\overline{l},0,0\right):\overline{D}_{\overline{l}}D_0D_0\hat{\Omega}_4=0;\\
\\
(\overline{a},b,c)=\left(\overline{l},0,i\right):\overline{D}_{\overline{l}}D_0D_i\hat{\Omega}_4=g_{i\overline{l}}D_0\hat{\Omega}_4;\\
\\
\quad(\overline{a},b,c)=\left(\overline{l},i,j\right):\\
\quad\overline{D}_{\overline{l}}D_iD_j\hat{\Omega}_4=iC_{ijk}g^{k\overline{k}}\hat{\Omega}_1\wedge\overline{D}_{\overline{l}}\overline{D}_{\overline{k}}\overline{\hat{\Omega}}_3=\\
\quad=g^{k\overline{k}}C_{ijk}\overline{C}_{\overline{l}\overline{m}\overline{k}}g^{m\overline{m}}\hat{\Omega}_1\wedge D_m\hat{\Omega}_3=\\
\quad=g^{k\overline{k}}C_{ijk}\overline{C}_{\overline{l}\overline{m}\overline{k}}g^{m\overline{m}}D_m\hat{\Omega}_4=\\
\quad\overset{SKG\ constraints}{=}\left(R_{i\overline{l}j\overline{m}}g^{m\overline{m}}+\delta_j^m g_{i\overline{l}}+\delta_i^m g_{j\overline{l}}\right)D_m\hat{\Omega}_4.
\end{array}\right. \tag{I.4}$$

Since the covariant derivatives of $\hat{\Omega}_4$ are often considered in local "flat" coordinated in M, below we write the independent ones, up to the third order included, by recalling Eqs. (4.1.1.8) and (4.1.1.11) (implemented by Eqs. (4.1.1.13)), Eqs. (4.1.1.5) and (4.1.1.12), and Eqs. (I.1)-(I.4):

$$D_A\hat{\Omega}_4:\left\{\begin{array}{l}
A=\underline{0}:D_{\underline{0}}\hat{\Omega}_4=e_{\underline{0}}^aD_a\hat{\Omega}_4=e_{\underline{0}}^0\left(D_0\hat{\Omega}_1\right)\wedge\hat{\Omega}_3=\overline{\hat{\Omega}}_1\wedge\hat{\Omega}_3;\\
\\
A=I:D_I\hat{\Omega}_4=e_I^aD_a\hat{\Omega}_4=e_I^iD_i\hat{\Omega}_4=e_I^i\left(\hat{\Omega}_1\wedge D_i\hat{\Omega}_3\right).
\end{array}\right. \tag{I.5}$$

$$D_A D_B \hat{\Omega}_4 = D_{(A} D_{B)} \hat{\Omega}_4 :$$

$$\begin{cases} (A,B) = (\underline{0},\underline{0}) : D_{\underline{0}} D_{\underline{0}} \hat{\Omega}_4 = e^a_{\underline{0}} e^b_{\underline{0}} D_a D_b \hat{\Omega}_4 = \left(e^0_{\underline{0}}\right)^2 D_0 D_0 \hat{\Omega}_4 = 0; \\ \\ (A,B) = (\underline{0},I) : D_{\underline{0}} D_I \hat{\Omega}_4 = e^a_{\underline{0}} e^b_I D_a D_b \hat{\Omega}_4 = e^0_{\underline{0}} e^i_I D_0 D_i \hat{\Omega}_4 = D_{\underline{0}} \hat{\Omega}_1 \wedge D_I \hat{\Omega}_3 = \overline{\hat{\Omega}}_1 \wedge D_I \hat{\Omega}_3; \\ \\ (A,B) = (I,J) : \begin{cases} D_I D_J \hat{\Omega}_4 = e^a_I e^b_J D_a D_b \hat{\Omega}_4 = \\ \\ = e^i_I e^j_J D_i D_j \hat{\Omega}_4 = e^i_I e^j_J \left(\hat{\Omega}_1 \wedge D_i D_j \hat{\Omega}_3\right) = \\ \\ = -e^i_I e^j_J e^{-K_1} C_{ijk} g^{k\bar{l}} \overline{D}_{\bar{0}} \overline{D}_{\bar{l}} \overline{\hat{\Omega}}_4 = i C_{IJK} \delta^{K\overline{K}} \hat{\Omega}_1 \wedge \overline{D}_{\overline{K}} \overline{\hat{\Omega}}_3 = \\ \\ = i C_{IJK} \delta^{K\overline{K}} \overline{D}_{\underline{\bar{0}}} \overline{\hat{\Omega}}_1 \wedge \overline{D}_{\overline{K}} \overline{\hat{\Omega}}_3. \end{cases} \end{cases} \tag{I.6}$$

$$D_A D_B D_C \hat{\Omega}_4 = D_{(A} D_B D_{C)} \hat{\Omega}_4 :$$

$$\begin{cases} (A,B,C) = (\underline{0},\underline{0},\underline{0}) : D_{\underline{0}} D_{\underline{0}} D_{\underline{0}} \hat{\Omega}_4 = e^a_{\underline{0}} e^b_{\underline{0}} e^c_{\underline{0}} D_a D_b D_c \hat{\Omega}_4 = \left(e^0_{\underline{0}}\right)^3 D_0 D_0 D_0 \hat{\Omega}_4 = 0; \\ \\ (A,B,C) = (\underline{0},\underline{0},I) : D_{\underline{0}} D_{\underline{0}} D_I \hat{\Omega}_4 = e^a_{\underline{0}} e^b_{\underline{0}} e^c_I D_a D_b D_c \hat{\Omega}_4 = \left(e^0_{\underline{0}}\right)^2 e^i_I D_0 D_0 D_i \hat{\Omega}_4 = 0; \\ \\ (A,B,C) = (\underline{0},I,J) : \begin{cases} D_{\underline{0}} D_I D_J \hat{\Omega}_4 = e^a_{\underline{0}} e^b_I e^c_J D_a D_b D_c \hat{\Omega}_4 = e^0_{\underline{0}} e^i_I e^j_J D_0 D_i D_j \hat{\Omega}_4 = \\ \\ = i e^0_{\underline{0}} e^i_I e^j_J \left(\bar{t}^{\bar{0}} - t^0\right)^{-1} C_{ijk} g^{k\bar{l}} \overline{D}_{\bar{l}} \overline{\hat{\Omega}}_4 = i C_{IJK} \delta^{K\overline{K}} \overline{D}_{\overline{K}} \overline{\hat{\Omega}}_4; \end{cases} \\ \\ (A,B,C) = (I,J,K) : \begin{cases} D_I D_J D_K \hat{\Omega}_4 = e^a_I e^b_J e^c_K D_a D_b D_c \hat{\Omega}_4 = e^i_I e^j_J e^k_K D_i D_j D_k \hat{\Omega}_4 = \\ \\ = -i e^i_I e^j_J e^k_K \left(\bar{t}^{\bar{0}} - t^0\right) \left(D_i C_{jkl}\right) g^{l\overline{m}} \overline{D}_{\bar{0}} \overline{D}_{\overline{m}} \overline{\hat{\Omega}}_4 + \\ \\ -i e^i_I e^j_J e^k_K \left(\bar{t}^{\bar{0}} - t^0\right) C_{ijk} \overline{D}_{\bar{0}} \overline{\hat{\Omega}}_4 = \\ \\ = i \left(D_I C_{JKL}\right) \delta^{L\overline{L}} \overline{D}_{\underline{\bar{0}}} \overline{D}_{\overline{L}} \overline{\hat{\Omega}}_4 + i C_{IJK} \overline{D}_{\underline{\bar{0}}} \overline{\hat{\Omega}}_4. \end{cases} \end{cases} \tag{I.7}$$

$$\overline{D}_{\overline{A}} D_B D_C \hat{\Omega}_4 = \overline{D}_{\overline{A}} D_{(B} D_{C)} \hat{\Omega}_4 :$$

$$\left\{ \begin{array}{l}
\left(\overline{A}, B, C\right) = \left(\underline{\overline{0}}, \underline{0}, \underline{0}\right) : \overline{D}_{\underline{\overline{0}}} D_{\underline{0}} D_{\underline{0}} \hat{\Omega}_4 = \overline{e}^{\overline{a}}_{\underline{\overline{0}}} e^{b}_{\underline{0}} e^{c}_{\underline{0}} \overline{D}_{\overline{a}} D_b D_c \hat{\Omega}_4 = \overline{e}^{\overline{0}}_{\underline{\overline{0}}} \left(e^{0}_{\underline{0}}\right)^2 \overline{D}_{\overline{0}} D_0 D_0 \hat{\Omega}_4 = 0; \\[2em]
\quad \left(\overline{A}, B, C\right) = \left(\underline{\overline{0}}, \underline{0}, I\right) : \overline{D}_{\underline{\overline{0}}} D_{\underline{0}} D_I \hat{\Omega}_4 = \overline{e}^{\overline{a}}_{\underline{\overline{0}}} e^{b}_{\underline{0}} e^{c}_{I} \overline{D}_{\overline{a}} D_b D_c \hat{\Omega}_4 = \\[1em]
\quad = \overline{e}^{\overline{0}}_{\underline{\overline{0}}} e^{0}_{\underline{0}} e^{i}_{I} \overline{D}_{\overline{0}} D_0 D_i \hat{\Omega}_4 = \overline{e}^{\overline{0}}_{\underline{\overline{0}}} e^{0}_{\underline{0}} e^{i}_{I} g_{0\overline{0}} D_i \hat{\Omega}_4 = D_I \hat{\Omega}_4; \\[2em]
\left(\overline{A}, B, C\right) = \left(\underline{\overline{0}}, I, J\right) : \overline{D}_{\underline{\overline{0}}} D_I D_J \hat{\Omega}_4 = \overline{e}^{\overline{a}}_{\underline{\overline{0}}} e^{b}_{I} e^{c}_{J} \overline{D}_{\overline{a}} D_b D_c \hat{\Omega}_4 = \overline{e}^{\overline{0}}_{\underline{\overline{0}}} e^{i}_{I} e^{j}_{J} \overline{D}_{\overline{0}} D_i D_j \hat{\Omega}_4 = 0; \\[2em]
\left(\overline{A}, B, C\right) = \left(\overline{L}, \underline{0}, \underline{0}\right) : \overline{D}_{\overline{L}} D_{\underline{0}} D_{\underline{0}} \hat{\Omega}_4 = \overline{e}^{\overline{a}}_{\overline{L}} e^{b}_{\underline{0}} e^{c}_{\underline{0}} \overline{D}_{\overline{a}} D_b D_c \hat{\Omega}_4 = \overline{e}^{\overline{l}}_{\overline{L}} \left(e^{0}_{\underline{0}}\right)^2 \overline{D}_{\overline{l}} D_0 D_0 \hat{\Omega}_4 = 0; \\[2em]
\quad \left(\overline{A}, B, C\right) = \left(\overline{L}, \underline{0}, I\right) : \overline{D}_{\overline{L}} D_{\underline{0}} D_I \hat{\Omega}_4 = \overline{e}^{\overline{a}}_{\overline{L}} e^{b}_{\underline{0}} e^{c}_{I} \overline{D}_{\overline{a}} D_b D_c \hat{\Omega}_4 = \\[1em]
\quad = \overline{e}^{\overline{l}}_{\overline{L}} e^{0}_{\underline{0}} e^{i}_{I} \overline{D}_{\overline{l}} D_0 D_i \hat{\Omega}_4 = \overline{e}^{\overline{l}}_{\overline{L}} e^{0}_{\underline{0}} e^{i}_{I} g_{i\overline{l}} D_0 \hat{\Omega}_4 = \delta_{I\overline{L}} D_{\underline{0}} \hat{\Omega}_4; \\[2em]
\left(\overline{A}, B, C\right) = \left(\overline{L}, I, J\right) : \left\{ \begin{array}{l}
\overline{D}_{\overline{L}} D_I D_J \hat{\Omega}_4 = \overline{e}^{\overline{a}}_{\overline{L}} e^{b}_{I} e^{c}_{J} \overline{D}_{\overline{a}} D_b D_c \hat{\Omega}_4 = \overline{e}^{\overline{l}}_{\overline{L}} e^{i}_{I} e^{j}_{J} \overline{D}_{\overline{l}} D_i D_j \hat{\Omega}_4 = \\[1em]
= \overline{e}^{\overline{l}}_{\overline{L}} e^{i}_{I} e^{j}_{J} g^{k\overline{k}} C_{ijk} \overline{C}_{\overline{l}\overline{m}\overline{k}} g^{m\overline{m}} D_m \hat{\Omega}_4 = \\[1em]
\overset{\textit{SKG constraints in local “flat” coords.}}{=} \left(\delta^{M}_{J} \delta_{I\overline{L}} + \delta^{M}_{I} \delta_{J\overline{L}}\right) D_M \hat{\Omega}_4.
\end{array} \right.
\end{array} \right. \qquad \text{(I.8)}$$

B Appendix II

The *“intersections”* among the elements of the set of 4-forms $\hat{\Omega}_4$, $D_0\hat{\Omega}_4$, $D_i\hat{\Omega}_4$, $D_0D_i\hat{\Omega}_4$, $\overline{\hat{\Omega}}_4$, $\overline{D}_{\overline{0}}\overline{\hat{\Omega}}_4$, $\overline{D}_{\overline{i}}\overline{\hat{\Omega}}_4$ and $\overline{D}_{\overline{0}}\overline{D}_{\overline{i}}\overline{\hat{\Omega}}_4$ in generic local “curved” and in local “flat” coordinates of M respectively read as follows:

$$\begin{gathered}\int_{CY_4}\hat{\Omega}_4\wedge\hat{\Omega}_4=0,\quad \int_{CY_4}\hat{\Omega}_4\wedge D_0\hat{\Omega}_4=0,\quad \int_{CY_4}\hat{\Omega}_4\wedge D_i\hat{\Omega}_4=0,\quad \int_{CY_4}\hat{\Omega}_4\wedge D_0D_i\hat{\Omega}_4=0;\\ \int_{CY_4}\hat{\Omega}_4\wedge\overline{\hat{\Omega}}_4=1;\end{gathered}\tag{II.1}$$

$$\begin{gathered}\int_{CY_4}\hat{\Omega}_4\wedge\overline{D}_{\overline{0}}\overline{\hat{\Omega}}_4=0,\quad \int_{CY_4}\hat{\Omega}_4\wedge\overline{D}_{\overline{i}}\overline{\hat{\Omega}}_4=0,\quad \int_{CY_4}\hat{\Omega}_4\wedge\overline{D}_{\overline{0}}\overline{D}_{\overline{i}}\overline{\hat{\Omega}}_4=0;\\ \int_{CY_4}D_i\hat{\Omega}_4\wedge D_j\hat{\Omega}_4=0,\quad \int_{CY_4}D_i\hat{\Omega}_4\wedge D_0\hat{\Omega}_4=0,\quad \int_{CY_4}D_i\hat{\Omega}_4\wedge D_0D_j\hat{\Omega}_4=0;\\ \int_{CY_4}D_i\hat{\Omega}_4\wedge\overline{D}_{\overline{j}}\overline{\hat{\Omega}}_4=-g_{i\overline{j}};\end{gathered}\tag{II.2}$$

$$\begin{gathered}\int_{CY_4}D_i\hat{\Omega}_4\wedge\overline{D}_{\overline{0}}\overline{\hat{\Omega}}_4=0,\quad \int_{CY_4}D_i\hat{\Omega}_4\wedge\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{\hat{\Omega}}_4=0;\\ \int_{CY_4}D_0\hat{\Omega}_4\wedge D_0\hat{\Omega}_4=0,\quad \int_{CY_4}D_0\hat{\Omega}_4\wedge D_0D_i\hat{\Omega}_4=0;\\ \int_{CY_4}D_0\hat{\Omega}_4\wedge\overline{D}_{\overline{0}}\overline{\hat{\Omega}}_4=-e^{2K_1}=\left(\overline{t}^{\overline{0}}-t^0\right)^{-2};\end{gathered}\tag{II.3}$$

$$\begin{gathered}\int_{CY_4}D_0\hat{\Omega}_4\wedge\overline{D}_{\overline{0}}\overline{D}_{\overline{i}}\overline{\hat{\Omega}}_4=0;\\ \int_{CY_4}D_0D_i\hat{\Omega}_4\wedge D_0D_j\hat{\Omega}_4=0;\\ \int_{CY_4}D_0D_i\hat{\Omega}_4\wedge\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{\hat{\Omega}}_4=e^{2K_1}g_{i\overline{j}}=-\left(\overline{t}^{\overline{0}}-t^0\right)^{-2}g_{i\overline{j}}.\end{gathered}\tag{II.4}$$

$$\begin{gathered}\int_{CY_4}\hat{\Omega}_4\wedge\hat{\Omega}_4=0,\quad \int_{CY_4}\hat{\Omega}_4\wedge D_{\underline{0}}\hat{\Omega}_4=0,\quad \int_{CY_4}\hat{\Omega}_4\wedge D_I\hat{\Omega}_4=0,\quad \int_{CY_4}\hat{\Omega}_4\wedge D_{\underline{0}}D_I\hat{\Omega}_4=0;\\ \int_{CY_4}\hat{\Omega}_4\wedge\overline{\hat{\Omega}}_4=1;\end{gathered}\tag{II.5}$$

$$\begin{gathered}\int_{CY_4}\hat{\Omega}_4\wedge\overline{D}_{\underline{\overline{0}}}\overline{\hat{\Omega}}_4=0,\quad \int_{CY_4}\hat{\Omega}_4\wedge\overline{D}_{\overline{I}}\overline{\hat{\Omega}}_4=0,\quad \int_{CY_4}\hat{\Omega}_4\wedge\overline{D}_{\underline{\overline{0}}}\overline{D}_{\overline{I}}\overline{\hat{\Omega}}_4=0;\\ \int_{CY_4}D_I\hat{\Omega}_4\wedge D_J\hat{\Omega}_4=0,\quad \int_{CY_4}D_I\hat{\Omega}_4\wedge D_{\underline{0}}\hat{\Omega}_4=0,\quad \int_{CY_4}D_I\hat{\Omega}_4\wedge D_{\underline{0}}D_J\hat{\Omega}_4=0;\\ \int_{CY_4}D_I\hat{\Omega}_4\wedge\overline{D}_{\overline{J}}\overline{\hat{\Omega}}_4=-e_I^i\overline{e}_{\overline{J}}^{\overline{j}}g_{i\overline{j}}=-\delta_{I\overline{J}};\end{gathered}\tag{II.6}$$

$$\begin{gathered}\int_{CY_4}D_I\hat{\Omega}_4\wedge\overline{D}_{\underline{\overline{0}}}\overline{\hat{\Omega}}_4=0,\quad \int_{CY_4}D_I\hat{\Omega}_4\wedge\overline{D}_{\underline{\overline{0}}}\overline{D}_{\overline{J}}\overline{\hat{\Omega}}_4=0;\\ \int_{CY_4}D_{\underline{0}}\hat{\Omega}_4\wedge D_{\underline{0}}\hat{\Omega}_4=0,\quad \int_{CY_4}D_{\underline{0}}\hat{\Omega}_4\wedge D_{\underline{0}}D_I\hat{\Omega}_4=0;\\ \int_{CY_4}D_{\underline{0}}\hat{\Omega}_4\wedge\overline{D}_{\underline{\overline{0}}}\overline{\hat{\Omega}}_4=-\left|e_{\underline{0}}^0\right|^2e^{2K_1}=-1;\end{gathered}\tag{II.7}$$

$$\begin{gathered}\int_{CY_4}D_{\underline{0}}\hat{\Omega}_4\wedge\overline{D}_{\underline{\overline{0}}}\overline{D}_{\overline{I}}\overline{\hat{\Omega}}_4=0;\\ \int_{CY_4}D_{\underline{0}}D_I\hat{\Omega}_4\wedge D_{\underline{0}}D_J\hat{\Omega}_4=0;\\ \int_{CY_4}D_{\underline{0}}D_I\hat{\Omega}_4\wedge\overline{D}_{\underline{\overline{0}}}\overline{D}_{\overline{J}}\overline{\hat{\Omega}}_4=\left|e_{\underline{0}}^0\right|^2e_I^i\overline{e}_{\overline{J}}^{\overline{j}}e^{2K_1}g_{i\overline{j}}=\delta_{I\overline{J}}.\end{gathered}\tag{II.8}$$

C Appendix III

The complete Hodge-decomposition of the real, Kähler gauge-invariant 4-form $\mathfrak{F}_4$ of Type IIB on $\frac{CY_3\times T^2}{\mathbb{Z}_2}$ in generic local "curved" coordinates in M reads as follows:

$$
\begin{aligned}
\mathfrak{F}_4 &= \left[\begin{array}{l} Z\overline{\hat{\Omega}}_4 - g^{a\overline{b}}\left(D_aZ\right)\overline{D}_{\overline{b}}\overline{\hat{\Omega}}_4 + \left|e^0_{\underline{0}}\right|^2 g^{a\overline{b}}\left(D_0D_aZ\right)\overline{D}_{\overline{0}}\overline{D}_{\overline{b}}\overline{\hat{\Omega}}_4 + \\ \\ + \left|e^0_{\underline{0}}\right|^2 g^{b\overline{a}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{a}}\overline{Z}\right)D_0D_b\hat{\Omega}_4 - g^{b\overline{a}}\left(\overline{D}_{\overline{a}}\overline{Z}\right)D_b\hat{\Omega}_4 + \overline{Z}\hat{\Omega}_4 \end{array}\right] = \\
&= 2Re\left[\overline{Z}\hat{\Omega}_4 - g^{a\overline{b}}\left(\overline{D}_{\overline{b}}\overline{Z}\right)D_a\hat{\Omega}_4 + \left|e^0_{\underline{0}}\right|^2 g^{a\overline{b}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{b}}\overline{Z}\right)D_0D_a\hat{\Omega}_4\right] = \qquad \text{(III.1)}\\
&= 2Re\left[\begin{array}{l} \overline{Z}\hat{\Omega}_1\wedge\hat{\Omega}_3 + \\ \\ + \left(t^0-\overline{t}^{\overline{0}}\right)^2\left(\overline{D}_{\overline{0}}\overline{Z}\right)\overline{\hat{\Omega}}_1\wedge\hat{\Omega}_3 - g^{i\overline{j}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)\hat{\Omega}_1\wedge D_i\hat{\Omega}_3 + \\ \\ + \left(t^0-\overline{t}^{\overline{0}}\right)g^{i\overline{j}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{Z}\right)\overline{\hat{\Omega}}_1\wedge D_i\hat{\Omega}_3 \end{array}\right] = \qquad \text{(III.2)}\\
&= 2Re\left[\begin{array}{l} \overline{Z}\hat{\Omega}_1\wedge\hat{\Omega}_3 + \\ \\ - \left|e^0_{\underline{0}}\right|^2\left(\overline{D}_{\overline{0}}\overline{Z}\right)\overline{\hat{\Omega}}_1\wedge\hat{\Omega}_3 - e^i_I\overline{e}^{\overline{j}}_{\overline{J}}\delta^{I\overline{J}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)\hat{\Omega}_1\wedge D_i\hat{\Omega}_3 + \\ \\ + \overline{e}^{\overline{0}}_{\underline{0}}e^i_I\overline{e}^{\overline{j}}_{\overline{J}}\delta^{I\overline{J}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{Z}\right)\overline{\hat{\Omega}}_1\wedge D_i\hat{\Omega}_3 \end{array}\right] = \qquad \text{(III.3)}\\
&= 2e^{K_1+K_3}Re\left[\begin{array}{l} \overline{W}\Omega_1\wedge\Omega_3 + \\ \\ - \left|e^0_{\underline{0}}\right|^2\left(\overline{D}_{\overline{0}}\overline{W}\right)\overline{\Omega}_1\wedge\Omega_3 - e^i_I\overline{e}^{\overline{j}}_{\overline{J}}\delta^{I\overline{J}}\left(\overline{D}_{\overline{j}}\overline{W}\right)\Omega_1\wedge D_i\Omega_3 + \\ \\ + \overline{e}^{\overline{0}}_{\underline{0}}e^i_I\overline{e}^{\overline{j}}_{\overline{J}}\delta^{I\overline{J}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{W}\right)\overline{\Omega}_1\wedge D_i\Omega_3 \end{array}\right]. \qquad \text{(III.4)}
\end{aligned}
$$

The evaluation of such identities along the constraints (4.2.6) yields the supersymmetric FV AEs in $\mathcal{N}=1$, $d=4$ supergravity from Type IIB on $\frac{CY_3\times T^2}{\mathbb{Z}_2}$ in local "curved" coordinates:

$$
\begin{aligned}
\mathfrak{F}_4 &= \left[Z\overline{\hat{\Omega}}_4 + \left|e^0_{\underline{0}}\right|^2 g^{a\overline{b}}\left(D_0D_aZ\right)\overline{D}_{\overline{0}}\overline{D}_{\overline{b}}\overline{\hat{\Omega}}_4 + \left|e^0_{\underline{0}}\right|^2 g^{b\overline{a}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{a}}\overline{Z}\right)D_0D_b\hat{\Omega}_4 + \overline{Z}\hat{\Omega}_4\right]_{SUSY} = \\
&= 2Re\left[\overline{Z}\hat{\Omega}_4 + \left|e^0_{\underline{0}}\right|^2 g^{a\overline{b}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{b}}\overline{Z}\right)D_0D_a\hat{\Omega}_4\right]_{SUSY} = \\
&= 2Re\left[\overline{Z}\hat{\Omega}_1\wedge\hat{\Omega}_3 + \left(t^0-\overline{t}^{\overline{0}}\right)g^{i\overline{j}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{Z}\right)\overline{\hat{\Omega}}_1\wedge D_i\hat{\Omega}_3\right]_{SUSY} = \\
&= 2Re\left[\overline{Z}\hat{\Omega}_1\wedge\hat{\Omega}_3 + \overline{e}^{\overline{0}}_{\underline{0}}e^i_I\overline{e}^{\overline{j}}_{\overline{J}}\delta^{I\overline{J}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{Z}\right)\overline{\hat{\Omega}}_1\wedge D_i\hat{\Omega}_3\right]_{SUSY} = \\
&= 2e^{K_1+K_3}Re\left[\overline{W}\Omega_1\wedge\Omega_3 + \overline{e}^{\overline{0}}_{\underline{0}}e^i_I\overline{e}^{\overline{j}}_{\overline{J}}\delta^{I\overline{J}}\left(\overline{D}_{\overline{0}}\overline{D}_{\overline{j}}\overline{W}\right)\overline{\Omega}_1\wedge D_i\Omega_3\right]_{SUSY}.
\end{aligned}
\qquad \text{(III.5)}
$$

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